A Calculus for Worms

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The Reflection Calculus

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$$\top \qquad \langle 1\rangle\langle 0\rangle\top \wedge \langle 5\rangle\top \qquad \langle 7\rangle(\langle 2\rangle\top \wedge \langle 0\rangle\langle 0\rangle\langle 0\rangle\top)$$

The Reflection Calculus: axioms and rules

Let φ, ψ, χ be formulas of RC^0_Λ, and $\alpha, \beta < \Lambda$.

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What are worms?

Worms are ~~long cylindrical animals with a tube-like body and no limbs~~ iterated consistency statements:

$$A = \langle 73 \rangle \langle \epsilon_0 \rangle \langle 0 \rangle \langle 42 \rangle \langle \omega \rangle \top$$

No one likes to write all of these $\langle$s and $\rangle$s...

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Decomposing worms

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$$A = h_\alpha(A) r_\alpha(A) \equiv_{\mathrm{RC}} h_\alpha(A) \wedge r_\alpha(A)$$

Why are worms interesting?

For any worms A, B such that $\min A > \alpha$,

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Theorem (Worms are the "core" of RC^0)

For each formula φ of $\mathsf{RC}^0_\wedge$ there is some worm A such that:

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Then maybe we could forget about conjunctions?

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When restricting the language to worms, the calculi are equivalent

Theorem. For any two worms A and D,

$$A \vdash_{\mathsf{RC}} D \text{ if and only if } A \vdash_{\mathsf{WC}} D.$$

Proof. It's easy to see that if $A \vdash_{\mathsf{WC}} D$ then $A \vdash_{\mathsf{RC}} D$.
...

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- Base case: $\mathit{length}(A\alpha B) = 1$.

$$\top \vdash_{\mathsf{RC}} \alpha \implies \top \vdash_{\mathsf{WC}} \alpha$$

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$$\top \vdash_{\mathsf{RC}} \alpha \implies \top \vdash_{\mathsf{WC}} \alpha$$

- Induction hypothesis: For any worms E, F such that $\mathit{length}(EF) < \mathit{length}(A\alpha B)$,

$$E \vdash_{\mathsf{RC}} F \implies E \vdash_{\mathsf{WC}} F$$

Conservativity: induction step

Goal: break down the statement

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into one or more smaller provability statements.
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For example:

$$\begin{array}{c} A \vdash \alpha B \\ {}_{\mathsf{RC}} \Downarrow \quad \Uparrow \\ h_\alpha(A) \vdash \alpha h_\alpha(B) \quad \text{and} \quad A \vdash r_\alpha(B) \end{array}$$

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And so on.

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- Modulo $\top \nvdash \alpha$, there is a syntactical algorithm to decide WC.
- Worms are great!

Thank you