On the strength of Hindman's Theorem for bounded sums of unions

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Outline

Hindman's Finite Sums Theorem

Theorem (Hindman, 1972)

Whenever the positive integers are colored in finitely many colors there is an infinite set such that all non-empty finite sums of distinct elements drawn from that set have the same color.

- Original proof is combinatorial but intricate.
- Later proofs are simpler but use strong methods (ultrafilters or ergodic theory).

Question, '80s

What is the strength of Hindman's Theorem?

Measures of Strength

$$\mathrm{HT}_k = \forall \underbrace{c : \mathbf{N} \to k}_{\text{instance}} \exists \underbrace{X \subseteq \mathbf{N}}_{\text{solution}} (|X| = \aleph_0 \text{ and } FS(X) \text{ is mono})$$

$$\mathrm{HT} = \forall k \mathrm{HT}_k$$

- Reverse Mathematics: **provability** in the systems

  $$\mathrm{RCA}_0, \mathrm{WKL}_0, \mathrm{ACA}_0, \mathrm{ACA}_0', \mathrm{ACA}_0^+, \ldots$$

  or (mutual) implications over the base theory RCA_0.
- Computable Mathematics: **complexity of solutions** for **computable instances**.
- RM and CM: computable **reducibility** to/from other principles.

Lower Bound on Hindman's Theorem

$$\mathrm{HT} \geq \varnothing^{(1)}, \mathrm{RT}^3_2, \mathrm{ACA}_0$$

Theorem (Blass, Hirst, Simpson 1987)

1. *Some computable (resp. computable in X) 2-coloring of* $\mathbf{N}$ *admits only solutions to* HT_2 *that compute* $\varnothing^{(1)}$ *(resp. X' – the jump of X).*
2. $\mathrm{RCA}_0 + \mathrm{HT}_2 \vdash \mathrm{ACA}_0$.

- Proof is by coding of the Halting Set and formalizes in RCA_0.
- Uses the notion of gap, the interval between two successive exponents of a number in base 2.

Upper Bound on Hindman's Theorem

$$\mathsf{ACA}_0^+, \varnothing^{(\omega+1)} \geq \mathsf{HT}$$

Theorem (Blass, Hirst, Simpson 1987)

1. *Any finite computable (resp. computable in X) coloring of* $\mathbf{N}$ *admits a solution to* HT *computable in* $\varnothing^{(\omega+1)}$ *(resp. in* $X^{(\omega+1)}$*).*
2. $\mathsf{ACA}_0^+ \vdash \mathsf{HT}$.

- ACA_0^+ is ACA_0 plus $\forall X \exists Y (Y = X^{(\omega)})$.
- Proof is by analyzing the original proof by Hindman.
- Ultrafilter and ergodic proofs give worse bounds (so far).

Bounded Sums

Question (Blass, 2005)

Does the complexity of HT grow with the length of the sums?

- Is it the case that longer sums require more jumps?
- $FS(X) =$ sums of finitely many distinct elements of X.
- $FS^{\leq n}(X) =$ sums of $1, 2, \ldots, n$ distinct elements of X.
- $\mathrm{HT}_k^{\leq n} =$ the restriction of HT to k colors and sums of length $\leq n$.

$$\mathrm{HT}_k^{\leq n}, \mathrm{HT}^{\leq n}$$

Lower Bounds for bounded sums

$$\mathsf{HT}^{\leq 3} \geq \emptyset^{(1)}, \mathsf{RT}^3_2, \mathsf{ACA}_0$$

Theorem (Dzhafarov, Jockusch, Solomon, Westrick, 2017)

1. $\mathsf{RCA}_0 + \mathsf{HT}^{\leq 3}_3 \vdash \mathsf{ACA}_0$.
2. $\mathsf{RCA}_0 \nvdash \mathsf{HT}^{\leq 2}_2$, *and* $\mathsf{RCA}_0 + \mathsf{RT}^1 + \mathsf{HT}^{\leq 2}_2 \vdash \mathsf{SRT}^2_2$.

- SRT^2_2 is the Stable Ramsey's Theorem ($\mathsf{WKL}_0 \nvdash \mathsf{SRT}^2_2$).
- Proof of (1): modification of Blass-Hirst-Simpson's argument.
- Proof of (2): Given a Δ^0_2-set A define a coloring all of whose solutions compute an infinite subset of A or an infinite set disjoint from A. Formalization requires RT^1 (eq. $B\Sigma^0_2$).

Upper bounds for bounded sums?

Question (Hindman, Leader and Strauss, 2003)

Is there a proof that whenever $\mathbf{N}$ is finitely coloured there is a sequence $x_1, x_2, \ldots$ such that all x_i and all $x_i + x_j$ ($i \neq j$) have the same colour, that does not also prove the Finite Sums Theorem?

- Does $\mathsf{HT}^{\leq 2}$ imply HT over RCA_0?
- Can we upper bound $\mathsf{HT}^{\leq 2}$ below ACA_0^+?
- Are there natural Hindman-type principles with:
  1. Non-trivial lower bounds, and
  2. Upper bounds strictly below HT?
- We call such principles Weak Yet Strong.

A brute force proof using Ramsey

Given $c : \mathbf{N} \to 2$,

1. Use RT^1_2 on $\mathbf{N}$ wrt c to get an infinite homset H_1.
2. Use RT^2_2 on H_1 wrt $f_2(x, y) := c(x + y)$ to fix the color of sums of length 2 on an infinite $H_2 \subseteq H_1$.

   ...

k. Use RT^k_2 on H_{k-1} wrt $f_k(x_1, \ldots, x_k) := c(x_1 + \cdots + x_k)$ to fix the color of sums of length k on an infinite $H_k \subseteq H_{k-1}$.

This induces a coloring $d : [1, k] \to 2$, where $d(i)$ is the c-color of sums of length i from H_k.

If k is large, then d has some interesting homogeneous set!

E.g. if $k \geq 6$ then by Schur's Theorem there exists $a, b > 0$ such that

$$d(a) = d(b) = d(a + b).$$

Hindman-Schur Theorem

- $FS^A(X)$: sums of j-many distinct elements of X for any $j \in A$.
- **Hindman-Schur Theorem**: Whenever the positive integers are colored in two colors **there exist** positive integers a, b and an infinite set H such that $FS^{\{a,b,a+b\}}(H)$ is monochromatic.

Theorem (C., 2017)

Hindman-Schur Theorem is provable in ACA_0.

- A host of similar Hindman-type theorems based on different finite combinatorial principles (e.g., Van Der Waerden, Folkman, etc.).
- All provable in ACA_0.
- What about lower bounds?

Hindman-Schur with apartness

The Blass-Hirst-Simpson's lower bound proof works, if we impose that the solution set satisfies the following Apartness Condition, for $t = 2$.

Definition (t-Apartness)

Fix a base $t \geq 2$. A set $X \subseteq \mathbf{N}$ satisfies the t-apartness condition if

$$x < x' \Rightarrow \mu_t(x) < \lambda_t(x').$$

$\lambda_t(x) =$ *least exponent in base t representation of n.*
$\mu_t(x) =$ *maximal exponent in base t representation of n.*

- P with t-apartness = P with t-apartness on the solution set.

Theorem (C., Kołodziejczyk, Lepore, Zdanowski, 2017)

Hindman-Schur with 2-apartness is equivalent to ACA_0 (over RCA_0).

The Apartness Condition

Imposing apartness is a **self-strenghtening** of Hindman's Theorem:

$$\mathsf{RCA}_0 \vdash \mathsf{HT} \equiv \mathsf{HT} \text{ with apartness.}$$

For restricted versions we have the following:

Proposition (C., Kołodziejczyk, Lepore, Zdanowski, 2017)

$\mathsf{RCA}_0 + \mathsf{HT}^{\leq n}_{2k} \vdash \mathsf{HT}^{\leq n}_{k}$ *with 3-apartness.*

Proof: Give $c : \mathbf{N} \to 2$, let $d : \mathbf{N} \to 4$:

$$d(n) := \begin{cases} c(n) & \text{if } n = 3^t + \dots, \\ 2 + c(n) & \text{if } n = 2 \cdot 3^t + \dots. \end{cases}$$

If $FS^{\leq 2}(H)$ is monochromatic for d then:

1. all elements have same first coefficient. Then:
2. no two elements of H can have the same first exponent.

Bounded Unions

Let $\mathsf{FUT}^{\leq n}_k$ and $\mathsf{FUT}^{=n}_k$ the versions of Hindman's Theorem in terms of unions instead of sums.

Proposition

For each $n, kt \geq 2$, $\mathsf{HT}^{\leq n}_k$ with t-apartness is equivalent to $\mathsf{FUT}^{\leq n}_k$ over RCA_0. Moreover, these principles are mutually strongly computably reducible. The same equivalences hold for $\mathsf{HT}^{=n}_k$ with t-apartness and $\mathsf{FUT}^{=n}_k$.

Restricted Hindman and Polarized Ramsey

Recall that Dzhafarov et alii proved

$$\mathsf{RCA}_0 + \mathsf{HT}^{\leq 2} + \mathsf{RT}^1 \vdash \mathsf{SRT}^2_2$$

We improve by showing that

$$\mathsf{RCA}_0 + \mathsf{HT}^{\leq 2} \vdash \mathsf{IPT}^2_2$$

Definition (Dzhafarov and Hirst, 2011)

IPT^2_2: *For all $f : [\mathbf{N}]^2 \to 2$ there exists a pair of infinite sets (H_1, H_2) such that all increasing pairs $\{x_1, x_2\}$ with $x_i \in H_i$ get the same f-color.*

$$\mathsf{RT}^2_2 \geq \mathsf{IPT}^2_2 > \mathsf{SRT}^2_2$$

Restricted Hindman and Polarized Ramsey

In fact we get that IPT^2_2 is **strongly computably reducible** to $\mathsf{HT}^{\leq 2}_4$: any $f : [\mathbf{N}]^2 \to 2$ of IPT^2_2 computes an instance $c : \mathbf{N} \to 2$ of $\mathsf{HT}^{\leq 2}_4$ s.t. any solution to $\mathsf{HT}^{\leq 2}_4$ for c computes a solution to IPT^2_2 for f.

Theorem (C., 2017)

$\mathsf{RCA}_0 + \mathsf{HT}^{\leq 2}_4 \vdash \mathsf{IPT}^2_2$. *Moreover,* $\mathsf{IPT}^2_2 \leq_{\mathrm{sc}} \mathsf{HT}^{\leq 2}_4$.

- $\mathsf{HT}^{=n}_k$ = restriction of HT_k to sums of *exactly* n elements.

In fact we show:

Theorem (C., 2017)

$\mathsf{RCA}_0 + \mathsf{HT}^{=2}_2$ *with t-apartness* $\vdash \mathsf{IPT}^2_2$. *Moreover,* $\mathsf{IPT}^2_2 \leq_{\mathrm{sc}} \mathsf{HT}^{=2}_2$ *with t-apartness.*

- N.B. RT^2_2 proves $\mathsf{HT}^{=2}_2$ with t-apartness.

$\mathsf{IPT}^2_2 \leq_{\mathrm{sc}} \mathsf{HT}^{=2}_2$ with 2-apartness

Given $f : [\mathbf{N}]^2 \to 2$, let $g : \mathbf{N} \to 2$:

$$g(n) := \begin{cases} 0 & \text{if } n = 2^t, \\ f(\lambda(n), \mu(n)) & \text{otherwise.} \end{cases}$$

Let $H = \{h_1 < h_2 < h_3 < \dots\}$ be an infinite and 2-apart set such that g is constant on $FS^{=2}(H)$. Then

$$\lambda(h_1) \leq \mu(h_1) < \lambda(h_2) \leq \mu(h_2) < \lambda(h_3) \leq \mu(h_3) < \dots$$

So if

$$H_1 := \{\lambda(h_1), \lambda(h_3), \lambda(h_5), \dots, \}$$

$$H_2 := \{\mu(h_2), \mu(h_4), \mu(h_6), \dots, \}$$

Then (H_1, H_2) is a solution to IPT^2_2 for f.

Sums of length 2 and ACA_0

$$\mathsf{HT}^{\leq 2} \geq \emptyset^{(1)}, \mathsf{RT}^3_2, \mathsf{ACA}_0$$

Recall that Dzhafarov et alii proved

$$\mathsf{RCA}_0 + \mathsf{HT}^{\leq 3} \vdash \mathsf{ACA}_0.$$

Theorem (C., Kołodziejczyk, Lepore, Zdanowski, 2017)

$\mathsf{RCA}_0 + \mathsf{HT}^{\leq 2} \vdash \mathsf{ACA}_0$.

Proposition (C., Kołodziejczyk, Lepore, Zdanowski, 2017)

For $t \geq 2$, $\mathsf{RCA}_0 + \mathsf{HT}^{\leq 2}_2$ with t-apartness $\vdash \mathsf{ACA}_0$.

$\mathrm{HT}_2^{\leq 2}$ with apartness implies ACA_0

Let $f : \mathbf{N} \to \mathbf{N}$ be 1:1. Let $n = 2^{n_0} + \cdots + 2^{n_r}$, $(n_0 < \cdots < n_r)$. Consider

$$f \restriction [0, n_0), f \restriction [n_0, n_1), \ldots, f \restriction [n_{r-1}, n_r).$$

Call $j \leq r$ **important in** n iff some value of $f \restriction [n_{j-1}, n_j)$ is below n_0. $(n_{-1} := 0)$.

$$c(n) := \text{ parity of the number of important } j\text{s in } n.$$

Let H be infinite, 2-apart and $\mathrm{FS}^{\leq 2}(H)$ mono.
Claim: for each $n \in H$ and each $x < \lambda(n)$,

$$x \in \mathrm{rg}(f) \text{ if and only if } x \in \mathrm{rg}(f \restriction \mu(n)).$$

Gives a computable definition of $\mathrm{rg}(f)$: given x, find the smallest $n \in H$ such that $x < \lambda(n)$ and check whether x is in $\mathrm{rg}(f \restriction \mu(n))$.

$\mathsf{HT}^{=n}_k$ with apartness and ACA_0

By improving the proof we get the following:

Proposition (C., Kołodziejczyk, Lepore, Zdanowski, 2017)

For every $t \geq 2$, $\mathsf{RCA}_0 + \mathsf{HT}^{=3}_2$ with t-apartness $\vdash \mathsf{ACA}_0$.

- Therefore $\{\mathsf{HT}^{=n}_k \text{ with 2-apartness } ; n \geq 3, k \geq 2\}$ is a weak yet strong family.

Corollary (C., Kołodziejczyk, Lepore, Zdanowski, 2017)

For every $n \geq 3$ and $k \geq 2$,

$$\mathsf{HT}^{=n}_k \textit{ with 2-apartness } \equiv \mathsf{ACA}_0$$

over RCA_0.

Open Problems

- Can we upper bound $\mathsf{HT}_2^{\leq 2}$ strictly below ACA_0^+?
- Is $\mathsf{HT}_2^{\leq 2}$ provable in ACA_0?
- Do colors matter? How?
- Does apartness increase strength in the bounded cases?
- Which implications are witnessed by reductions? E.g. Does $\mathsf{IPT}_2^3 \leq_{\mathrm{sc}} \mathsf{HT}_2^{\leq 3}$?

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