On the strength of Hindman’s Theorem for bounded sums of unions

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September 2017
Wormshop 2017
Moscow, Steklov Institute
Outline

1. Hindman’s Finite Sums Theorem
2. Bounded Sums
3. Weak Yet Strong Principles
4. From Hindman to Ramsey
5. Other variants
Hindman’s Finite Sums Theorem

**Theorem (Hindman, 1972)**

Whenever the positive integers are colored in finitely many colors there is an infinite set such that all non-empty finite sums of distinct elements drawn from that set have the same color.

- Original proof is combinatorial but intricate.
- Later proofs are simpler but use strong methods (ultrafilters or ergodic theory).

**Question, ’80s**

What is the strength of Hindman’s Theorem?
Measures of Strength

\[ HT_k = \forall c : \mathbb{N} \rightarrow k \exists X \subseteq \mathbb{N}(|X| = \aleph_0 \text{ and } FS(X) \text{ is mono}) \]

\[ HT = \forall k HT_k \]

- **Reverse Mathematics:** provability in the systems
  
  \[ \text{RCA}_0, \text{WKL}_0, \text{ACA}_0, \text{ACA}'_0, \text{ACA}^+_0, \ldots \]

  or (mutual) implications over the base theory \( \text{RCA}_0 \).

- **Computable Mathematics:** complexity of solutions for computable instances.

- **RM and CM:** computable reducibility to/from other principles.
Lower Bound on Hindman’s Theorem

\[ \text{HT} \geq \emptyset^{(1)}, \text{RT}_2^3, \text{ACA}_0 \]

**Theorem (Blass, Hirst, Simpson 1987)**

1. Some computable (resp. computable in \(X\)) 2-coloring of \(\mathbb{N}\) admits only solutions to \(\text{HT}_2\) that compute \(\emptyset^{(1)}\) (resp. \(X'\) – the jump of \(X\)).
2. \(\text{RCA}_0 + \text{HT}_2 \vdash \text{ACA}_0\).

- Proof is by coding of the Halting Set and formalizes in \(\text{RCA}_0\).
- Uses the notion of gap, the interval between two successive exponents of a number in base 2.
Upper Bound on Hindman’s Theorem

$$\text{ACA}^+_0, \emptyset^{(\omega+1)} \geq \text{HT}$$

Theorem (Blass, Hirst, Simpson 1987)

1. Any finite computable (resp. computable in $X$) coloring of $\mathbb{N}$ admits a solution to HT computable in $\emptyset^{(\omega+1)}$ (resp. in $X^{(\omega+1)}$).

2. $\text{ACA}^+_0 \vdash \text{HT}$.

- $\text{ACA}^+_0$ is $\text{ACA}_0$ plus $\forall X \exists Y (Y = X^{(\omega)})$.
- Proof is by analyzing the original proof by Hindman.
- Ultrafilter and ergodic proofs give worse bounds (so far).
Bounded Sums

Question (Blass, 2005)
Does the complexity of HT grow with the length of the sums?

- Is it the case that longer sums require more jumps?
- $FS(X) = \text{sums of finitely many distinct elements of } X$.
- $FS_{\leq n}(X) = \text{sums of } 1, 2, \ldots, n \text{ distinct elements of } X$.
- $HT_{\leq n}^k = \text{the restriction of HT to } k \text{ colors and sums of length } \leq n$.

$HT_{\leq n}^k, HT_{\leq n}$
Lower Bounds for bounded sums

\( \text{HT}^{\leq 3} \geq \emptyset^{(1)} , \text{RT}_{2}^{3} , \text{ACA}_{0} \)

Theorem (Dzhafarov, Jockusch, Solomon, Westrick, 2017)

1. \( \text{RCA}_{0} + \text{HT}_{3}^{\leq 3} \vdash \text{ACA}_{0} \).
2. \( \text{RCA}_{0} \nvdash \text{HT}_{2}^{\leq 2} \), and \( \text{RCA}_{0} + \text{RT}^{1} + \text{HT}_{2}^{\leq 2} \vdash \text{SRT}_{2}^{2} \).

- \( \text{SRT}_{2}^{2} \) is the Stable Ramsey’s Theorem (\( \text{WKL}_{0} \nvdash \text{SRT}_{2}^{2} \)).
- Proof of (2): Given a \( \Delta_{2}^{0} \)-set \( A \) define a coloring all of whose solutions compute an infinite subset of \( A \) or an infinite set disjoint from \( A \). Formalization requires \( \text{RT}^{1} \) (eq. \( \text{B}\Sigma_{2}^{0} \)).
Upper bounds for bounded sums?

Question (Hindman, Leader and Strauss, 2003)
Is there a proof that whenever $\mathbb{N}$ is finitely coloured there is a sequence $x_1, x_2, \ldots$ such that all $x_i$ and all $x_i + x_j \ (i \neq j)$ have the same colour, that does not also prove the Finite Sums Theorem?

- Does $HT^{\leq 2}$ imply $HT$ over $RCA_0$?
- Can we upper bound $HT^{\leq 2}$ below $ACA_0^+$?
- Are there natural Hindman-type principles with:
  1. Non-trivial lower bounds, and
  2. Upper bounds strictly below $HT$?
- We call such principles Weak Yet Strong.
A brute force proof using Ramsey

Given $c : \mathbb{N} \to 2$,

1. Use $RT_2^1$ on $\mathbb{N}$ wrt $c$ to get an infinite homset $H_1$.

2. Use $RT_2^2$ on $H_1$ wrt $f_2(x, y) := c(x + y)$ to fix the color of sums of length 2 on an infinite $H_2 \subseteq H_1$.

\ldots

k. Use $RT_2^k$ on $H_{k-1}$ wrt $f_k(x_1, \ldots, x_k) := c(x_1 + \cdots + x_k)$ to fix the color of sums of length $k$ on an infinite $H_k \subseteq H_{k-1}$.

This induces a coloring $d : [1, k] \to 2$, where $d(i)$ is the $c$-color of sums of length $i$ from $H_k$.

If $k$ is large, then $d$ has some interesting homogeneous set!

E.g. if $k \geq 6$ then by Schur’s Theorem there exists $a, b > 0$ such that

$$d(a) = d(b) = d(a + b).$$
Hindman-Schur Theorem

- $FS^A(X)$: sums of $j$-many distinct elements of $X$ for any $j \in A$.
- **Hindman-Schur Theorem**: Whenever the positive integers are colored in two colors there exist positive integers $a, b$ and an infinite set $H$ such that $FS\{a,b,a+b\}(H)$ is monochromatic.

**Theorem (C., 2017)**

*Hindman-Schur Theorem is provable in ACA$_0$.*

- A host of similar Hindman-type theorems based on different finite combinatorial principles (e.g., Van Der Waerden, Folkman, etc.).
- All provable in ACA$_0$.
- What about lower bounds?
Hindman-Schur with apartness

The Blass-Hirst-Simpson’s lower bound proof works, if we impose that the solution set satisfies the following Apartness Condition, for $t = 2$.

**Definition ($t$-Apartness)**

Fix a base $t \geq 2$. A set $X \subseteq \mathbb{N}$ satisfies the $t$-apartness condition if

$$x < x' \Rightarrow \mu_t(x) < \lambda_t(x').$$

- $\lambda_t(x) =$ least exponent in base $t$ representation of $n$.
- $\mu_t(x) =$ maximal exponent in base $t$ representation of $n$.

P with $t$-apartness = P with $t$-apartness on the solution set.

**Theorem (C., Kołodziejczyk, Lepore, Zdanowski, 2017)**

**Hindman-Schur with 2-apartness is equivalent to ACA$_0$ (over RCA$_0$).**
The Apartness Condition

Imposing apartness is a **self-strengthening** of Hindman’s Theorem:

\[ \text{RCA}_0 \vdash \text{HT} \equiv \text{HT with apartness}. \]

For restricted versions we have the following:

**Proposition (C., Kołodziejczyk, Lepore, Zdanowski, 2017)**

\[ \text{RCA}_0 + \text{HT}^\leq_{2k} \vdash \text{HT}^\leq_{k} \text{ with 3-apartness}. \]

**Proof:** Give \( c : \mathbb{N} \to 2 \), let \( d : \mathbb{N} \to 4 \):

\[
d(n) := \begin{cases} 
c(n) & \text{if } n = 3^t + \ldots, \\
2 + c(n) & \text{if } n = 2 \cdot 3^t + \ldots.
\end{cases}
\]

If \( FS^\leq_2(H) \) is monochromatic for \( d \) then:

1. all elements have same first coefficient. Then:
2. no two elements of \( H \) can have the same first exponent.
Let $\text{FUT}_k^{\leq n}$ and $\text{FUT}_k^=n$ the versions of Hindman’s Theorem in terms of unions instead of sums.

**Proposition**

For each $n$, $kt \geq 2$, $\text{HT}_k^{\leq n}$ with $t$-apartness is equivalent to $\text{FUT}_k^{\leq n}$ over $\text{RCA}_0$. Moreover, these principles are mutually strongly computably reducible. The same equivalences hold for $\text{HT}_k^=n$ with $t$-apartness and $\text{FUT}_k^=n$. 
Restricted Hindman and Polarized Ramsey

Recall that Dzhafarov et alii proved

$$\text{RCA}_0 + \text{HT}^{\leq 2} + \text{RT}^1 \vdash \text{SRT}^2_2$$

We improve by showing that

$$\text{RCA}_0 + \text{HT}^{\leq 2} \vdash \text{IPT}^2_2$$

Definition (Dzhafarov and Hirst, 2011)

**IPT**$^2_2$: For all $f : [\mathbb{N}]^2 \to 2$ there exists a pair of infinite sets $(H_1, H_2)$ such that all increasing pairs $\{x_1, x_2\}$ with $x_i \in H_i$ get the same $f$-color.

$$\text{RT}^2_2 \geq \text{IPT}^2_2 > \text{SRT}^2_2$$
Restricted Hindman and Polarized Ramsey

In fact we get that $\text{IPT}_2^2$ is strongly computably reducible to $\text{HT}_{4}^{\leq 2}$: any $f : [\mathbb{N}]^2 \to 2$ of $\text{IPT}_2^2$ computes an instance $c : \mathbb{N} \to 2$ of $\text{HT}_{4}^{\leq 2}$ s.t. any solution to $\text{HT}_{4}^{\leq 2}$ for $c$ computes a solution to $\text{IPT}_2^2$ for $f$.

**Theorem (C., 2017)**

$\text{RCA}_0 + \text{HT}_{4}^{\leq 2} \vdash \text{IPT}_2^2$. Moreover, $\text{IPT}_2^2 \leq_{\text{sc}} \text{HT}_{4}^{\leq 2}$.

- $\text{HT}_k^{=n} = \text{restriction of HT}_k$ to sums of exactly $n$ elements.

In fact we show:

**Theorem (C., 2017)**

$\text{RCA}_0 + \text{HT}_2^{=2} \mathbin{\text{with \ t-apartness}} \vdash \text{IPT}_2^2$. Moreover, $\text{IPT}_2^2 \leq_{\text{sc}} \text{HT}_2^{=2 \text{ with t-apartness}}$.

- N.B. $\text{RT}_2^2$ proves $\text{HT}_2^{=2}$ with t-apartness.
Given $f : [\mathbb{N}]^2 \to 2$, let $g : \mathbb{N} \to 2$:

$$g(n) := \begin{cases} 0 & \text{if } n = 2^t, \\ f(\lambda(n), \mu(n)) & \text{otherwise.} \end{cases}$$

Let $H = \{ h_1 < h_2 < h_3 < \ldots \}$ be an infinite and 2-apart set such that $g$ is constant on $FS^=^2(H)$. Then

$$\lambda(h_1) \leq \mu(h_1) < \lambda(h_2) \leq \mu(h_2) < \lambda(h_3) \leq \mu(h_3) < \ldots$$

So if

$$H_1 := \{ \lambda(h_1), \lambda(h_3), \lambda(h_5), \ldots, \}$$

$$H_2 := \{ \mu(h_2), \mu(h_4), \mu(h_6), \ldots, \}$$

Then $(H_1, H_2)$ is a solution to IPT$^2_2$ for $f$. 
Sums of length 2 and $\text{ACA}_0$

$$\text{HT}^{\leq 2} \geq \emptyset^{(1)}, \text{RT}^3_2, \text{ACA}_0$$

Recall that Dzhafarov et alii proved

$$\text{RCA}_0 + \text{HT}^{\leq 3} \vdash \text{ACA}_0.$$
$\text{HT}_{2}^{\leq 2}$ with apartness implies $\text{ACA}_0$

Let $f : \mathbb{N} \to \mathbb{N}$ be 1:1. Let $n = 2^{n_0} + \cdots + 2^{n_r}$, $(n_0 < \cdots < n_r)$. Consider

$$f \upharpoonright [0, n_0), f \upharpoonright [n_0, n_1), \ldots, f \upharpoonright [n_{r-1}, n_r).$$

Call $j \leq r$ important in $n$ iff some value of $f \upharpoonright [n_{j-1}, n_j)$ is below $n_0$. ($n_{-1} := 0$).

$$c(n) := \text{parity of the number of important } j \text{s in } n.$$

Let $H$ be infinite, 2-apart and $\text{FS}^{\leq 2}(H)$ mono. **Claim:** for each $n \in H$ and each $x < \lambda(n)$,

$$x \in \text{rg}(f) \text{ if and only if } x \in \text{rg}(f \upharpoonright \mu(n)).$$

Gives a computable definition of $\text{rg}(f)$: given $x$, find the smallest $n \in H$ such that $x < \lambda(n)$ and check whether $x$ is in $\text{rg}(f \upharpoonright \mu(n))$. 
HT\textsuperscript{=}\textsubscript{n} with apartness and ACA\textsubscript{0}

By improving the proof we get the following:

**Proposition (C., Kołodziejczyk, Lepore, Zdanowski, 2017)**

For every \( t \geq 2 \), RCA\textsubscript{0} + HT\textsuperscript{=}\textsubscript{3} with \( t \)-apartness \( \vdash \) ACA\textsubscript{0}.

Therefore \( \{ \text{HT}\textsubscript{k} with 2-apartness ; n \geq 3, k \geq 2 \} \) is a weak yet strong family.

**Corollary (C., Kołodziejczyk, Lepore, Zdanowski, 2017)**

For every \( n \geq 3 \) and \( k \geq 2 \),

\[
\text{HT}\textsubscript{k} with 2-apartness \equiv \text{ACA}_0
\]

over RCA\textsubscript{0}.
Open Problems

- Can we upper bound $HT_{2}^{\leq 2}$ strictly below $ACA_{0}^{+}$?
- Is $HT_{2}^{\leq 2}$ provable in $ACA_{0}$?
- Do colors matter? How?
- Does apartness increase strength in the bounded cases?
- Which implications are witnessed by reductions? E.g. Does $IPT_{2}^{3} \leq_{sc} HT_{2}^{\leq 3}$?


