Caristi’s fixed point theorem and strong systems of arithmetic

David Fernández-Duque
Mathematics Department, Ghent University
David.FernandezDuque@UGent.be

Joint with Paul Shafer, Henry Towsner, and Keita Yokoyama.
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**Theorem (Banach, 1922)**

Let $\mathcal{X}$ be a complete metric space and $f : \mathcal{X} \to \mathcal{X}$ be a contraction; that is, there is $\rho < 1$ such that $d(f(x), f(y)) < \rho \cdot d(x, y)$ for all $x, y \in X$. Then, there is $x_\ast \in X$ such that $f(x_\ast) = x_\ast$. 

**Theorem (Brouwer, 1910)**

Let $D$ be a disk in $\mathbb{R}^n$ and $f : D \to D$ be continuous. Then, there is $x_\ast \in X$ such that $f(x_\ast) = x_\ast$. 
Fixed point theorems in analysis

**Theorem (Banach, 1922)**

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Caristi’s theorem and strong arithmetics

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**Definition**

A **Caristi system** is a triple $(\mathcal{X}, V, f)$, where

- $\mathcal{X}$ is a complete separable metric space,
- $V: \mathcal{X} \to \mathbb{R}_{\geq 0}$ is a lower semi-continuous function, and
- $f: \mathcal{X} \to \mathcal{X}$ is an arbitrary function,

such that

$$\forall x \in \mathcal{X} \left( d(x, f(x)) \leq V(x) - V(f(x)) \right).$$

**Theorem (Caristi, 1976)**

*If $(\mathcal{X}, V, f)$ is a Caristi system, then $f$ has a fixed point.*

- Henceforth, a **metric space** is a complete separable metric space.
- We call the $V$ in a Caristi system a **potential**.
Proofs of Caristi’s theorem

1. Caristi’s proof (simplified by Chi Song Wong)
Proofs of Caristi’s theorem

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2. Proof by EVP.

Theorem (Ekeland, 1974)

Every lower semi-continuous function $V : X \to \mathbb{R}_{\geq 0}$ has a critical point, i.e. a point $x_\ast \in X$ such that

$$\forall y \in X \left( d(x_\ast, y) \leq V(x_\ast) - V(y) \rightarrow y = x_\ast \right)$$
The Big Five subsystems of second-order arithmetic

- **Two sorts:** natural numbers, sets of naturals.
- Real numbers, infinite trees, etc. can all be coded in SOA.
- All systems have $\Sigma^0_1$-induction, elementary arithmetical axioms.

\[ \text{RCA}_0 \quad \text{Computable sets exist (}\Delta^0_1\text{ comprehension)} \]

\[ \text{WKL}_0 \quad \text{RCA}_0 + \text{“every infinite binary tree has an infinite path”} \]

\[ \text{ACA}_0 \quad \{n \in \mathbb{N} : \varphi(n)\} \text{ exists, } \varphi \text{ arithmetical} \]

\[ \text{ATR}_0 \quad \text{Transfinite recursion for arithmetical formulas.} \]

\[ \Pi^1_1-\text{CA}_0 \quad \{n \in \mathbb{N} : \forall X \subseteq \mathbb{N} \, \varphi(n, X)\} \text{ exists, } \varphi \text{ arithmetical} \]
The strength of continuous Caristi’s theorem

**Theorem (F-D S T Y)**

*The following are equivalent over RCA₀.*

1. ACA₀.
2. Caristi’s theorem for continuous functions.
3. Caristi’s theorem for continuous potentials and continuous functions.

**Proof.** (1 → 2). By EVP.
The strength of continuous Caristi’s theorem

**Theorem (F-D S T Y)**

**The following are equivalent over RCA\(_0\).**

1. ACA\(_0\).
2. Caristi’s theorem for continuous functions.
3. Caristi’s theorem for continuous potentials and continuous functions.

**Proof.** (1 $\rightarrow$ 2). By EVP.

(3 $\rightarrow$ 1). Use the fact that ACA\(_0\) is equivalent to the statement that every decreasing sequence of positive reals has an infimum.
Compactness and Caristi’s theorem

**Theorem (F-D S T Y)**

*For compact metric spaces $X$:

- Caristi for l.s.c. $V$ and continuous $f$ is equivalent to $\text{WKL}_0$.
- Caristi for continuous $V$ and continuous $f$ is equivalent to $\text{WKL}_0$.***
Compactness and Caristi’s theorem

Theorem (F-D S T Y)

For compact metric spaces $\mathcal{X}$:

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Baire functions:

- Continuous functions are Baire class 0
- $\omega$-limits of Baire class $< \xi$ functions are Baire class $\xi$. 
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- Continuous functions are Baire class 0
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Theorem (F-D S T Y)

For compact metric spaces $\mathcal{X}$:

- Caristi for l.s.c. $V$ and Baire class 1 $f$ is equivalent to ACA$_0$. 

**Caristi vs. ATR\textsubscript{0}**

### Theorem (F-D S T Y)

*Caristi for Baire class 1 functions $f : X \to X$ (with arbitrary $X$ and l.s.c. $V$) implies ATR\textsubscript{0}.*

<table>
<thead>
<tr>
<th>Facts:</th>
</tr>
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<tbody>
<tr>
<td>1. $\phi$-TR := $\forall \prec \left( \text{WO}(\prec) \to \exists Z \forall \xi \forall n (n \in Z \iff \phi(n,Z \prec \xi)) \right)$</td>
</tr>
<tr>
<td>2. $\text{ATR}<em>{0} \equiv \text{RCA}</em>{0} + \Sigma_{0}^{1}$-TR</td>
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<td>3. If $\phi$ is $\Sigma_{0}^{1}$, there is a sequence of trees $(T_{\xi})<em>{\xi \in \mathbb{N}}$ such that any path $g$ through $T</em>{\xi}$ codes $Z_{\xi}$.</td>
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### Theorem (F-D S T Y)

**Caristi for Baire class 1 functions** $f : X \to X$ (with arbitrary $X$ and l.s.c. $V$) implies ATR$_0$.

### Facts:

1. \( \varphi\text{-}TR := \forall \prec \left( WO(\prec) \rightarrow \exists Z \ \forall \xi \ \forall n \ (n \in Z_\xi \iff \varphi(n, Z_\prec \xi)) \right) \)
   - \( Z_\xi = \{ m \in \mathbb{N} : \langle m, \xi \rangle \in Z \} \),
   - \( Z_\prec \xi = \{ m \in \mathbb{N} : \exists \zeta < \xi \langle m, \zeta \rangle \in Z \} \).
Caristi vs. ATR$_0$

**Theorem (F-D S T Y)**

*Caristi for Baire class 1 functions* $f : \mathcal{X} \rightarrow \mathcal{X}$ (with arbitrary $\mathcal{X}$ and l.s.c. $V$) implies ATR$_0$.

**Facts:**

1. $\varphi$-TR $:=$ $\forall \prec \left( WO(\prec) \rightarrow \exists Z \ \forall \xi \ \forall n \ (n \in Z_\xi \leftrightarrow \varphi(n, Z_{\prec\xi})) \right)$
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2. ATR$_0 \equiv$ RCA$_0 + \Sigma^0_1$-TR.
Caristi vs. $\text{ATR}_0$

**Theorem (F-D S T Y)**

*Caristi for Baire class 1 functions* $f : \mathcal{X} \to \mathcal{X}$ (with arbitrary $\mathcal{X}$ and l.s.c. $V$) implies $\text{ATR}_0$.

**Facts:**

1. $\varphi$-$\text{TR} := \forall \prec (WO(\prec) \to \exists Z \ \forall \xi \ \forall n \ (n \in Z_\xi \iff \varphi(n, Z_\prec \xi)))$
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2. $\text{ATR}_0 \equiv \text{RCA}_0 + \Sigma^0_1$-$\text{TR}$.

3. If $\varphi$ is $\Sigma^0_1$, there is a sequence of trees $(T_\xi)_{\xi \in \mathbb{N}}$ such that any path $g$ through $T_\xi$ codes $Z_\xi$.
Caristi for Baire class 1 $F$ implies $\text{ATR}_0$.

Proof idea. Fix $(T_i)_{i<\omega}$. We define a Caristi system $(\mathcal{X}, F, V)$ as follows.

- $\mathcal{X}$ be the set of sequences of paths $(g_i)_{i<\omega}$.

- $F(\vec{g}) = \lim_{n \to \omega} F_n$, where $F_n(\vec{g})$ replaces $g_\xi$ by an ‘$n$-approximation’ of the path through $T_\xi$, whenever:
  1. $g_\xi(n) \not\in T_\xi$, and
  2. $\xi < n$ is the $\prec$-minimum satisfying 1.

- $V(\vec{g}) = \sum \{2^{-n} : g_n$ is not a path through $T_i\}$.

This defines a Caristi system, whose fixed point $(g_i^*)_{i<\omega}$ such that each $g_i^*$ is a path through $T_i$. $\square$
Leftmost paths

**Theorem (Marcone)**

$\Pi^1_1$-CA$_0$ is equivalent to the statement “every ill-founded tree $T \subseteq \mathbb{N}^{<\mathbb{N}}$ has a leftmost path.”

**Definition (Towsner)**

- The **transfinite leftmost path principle** states that if $T \subseteq \mathbb{N}^{<\mathbb{N}}$ is ill-founded and $\alpha$ is a well-order, then there is a path $f^*$ through $T$ such that no path through $T$ is both $\Sigma^T_\alpha f^*$ and to the left of $f^*$.

- $\text{TLPP}_0$ is RCA$_0$ plus the transfinite leftmost path principle.

$\text{TLPP}_0$ is strictly between ATR$_0$ and $\Pi^1_1$-CA$_0$. 
Caristi’s fixed point theorem for Baire functions

Theorem (F-D S T Y)

Caristi for Baire functions \( f : \mathcal{X} \to \mathcal{X} \) (with arbitrary \( \mathcal{X} \) and l.s.c. \( V \)) is equivalent to TLPP\(_0\).

Thus in the general case:

- Caristi is equivalent to TLPP\(_0\)
- Ekeland is equivalent to \( \Pi^1_1\)-CA\(_0\)
Strength of Caristi’s proof

Recall: Caristi’s proof relies on uncountable Caristi sequences.

Definition

Fix a Caristi system \((X, V, f)\).

A Caristi sequence (from \(x_0\)) is a well-order \((L, \prec)\) and a sequence \((x_\ell : \ell \in L) \subseteq X\) such that

\[
\begin{align*}
  x_{\text{min} L} &= x_0 \\
  x_{S(\ell)} &= f(x_\ell) \\
  x_\ell &= \lim_{k<\ell} x_k \quad (\ell \in \text{Lim})
\end{align*}
\]
Maximal sequences

Call a Caristi sequence **proper** if the $x_\ell$’s are all distinct.

**Lemma (Maximal sequence principle)**

Given a Caristi system $(\mathcal{X}, V, f)$ with $f$ arithmetical and $x_0 \in \mathcal{X}$, there is a proper Caristi sequence with no strict, proper extensions.
Maximal sequences

Call a Caristi sequence **proper** if the $x_\ell$'s are all distinct.

**Lemma (Maximal sequence principle)**

Given a Caristi system $(X, V, f)$ with $f$ arithmetical and $x_0 \in X$, there is a proper Caristi sequence with no strict, proper extensions.

**Proof of Caristi’s theorem by MSP**: Any maximal sequence must have a last element, which is a fixed point of $f$. □
The closed orbit principle

Closed orbit: If \((\mathcal{X}, f)\) is a dynamical system and \(x \in \mathcal{X}\), the closed orbit of \(x\) is the least topologically closed, \(f\)-closed set \(O^*\) such that \(x \in O^*\).
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**Theorem (Banach fixed point theorem 2.0)**

If \(\mathcal{X}\) is a metric space and \(f : \mathcal{X} \to \mathcal{X}\) is a contraction, then for any \(x \in \mathcal{X}\), the closed orbit of \(\mathcal{X}\) has a unique fixed point.
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Theorem (Banach fixed point theorem 2.0)

If \(\mathcal{X}\) is a metric space and \(f: \mathcal{X} \rightarrow \mathcal{X}\) is a contraction, then for any \(x \in \mathcal{X}\), the closed orbit of \(\mathcal{X}\) has a unique fixed point.

Lemma (Closed orbit principle)

For every Caristi system \((\mathcal{X}, V, f)\) with \(f\) arithmetical and every \(x_0 \in \mathcal{X}\), there is a \(\subseteq\)-least closed set \(O^*\) such that \(x_0 \in O^*\) and
\[
(\forall x \in \mathcal{X})(x \in O^* \rightarrow f(x) \in O^*).
\]
Inflationary fixed points

Definition

The **arithmetical inflationary fixed point scheme** is the scheme stating that if \( F : 2^\mathbb{N} \to 2^\mathbb{N} \) is arithmetical and

\[
\forall X \ (X \subseteq F(X)),
\]

then there is a w.o. \((L, \prec)\) with max and sets \((X_\alpha : \alpha \in L)\) such that

\[
\begin{align*}
X_{\min L} &= \emptyset \\
X_{S(\alpha)} &= F(X_\alpha) \\
X_\gamma &= \bigcup_{\alpha \prec \gamma} X_\alpha \quad (\gamma \in \text{Lim}) \\
F(X_{\max L}) &= X_{\max L}.
\end{align*}
\]

**Stronger** than \(\Pi^1_1\text{-CA}_0\): does not require \(X \subseteq Y \rightarrow F(X) \subseteq F(Y)\)
Theorem (F-D S T Y)

*The arithmetical inflationary fixed point scheme*

≡ *the maximal sequence principle*

≡ *the closed orbit principle*
**Theorem (F-D S T Y)**

The arithmetical inflationary fixed point scheme

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<th>$f$</th>
<th>$V$</th>
<th>Caristi</th>
<th>Ekeland</th>
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</thead>
<tbody>
<tr>
<td>compact</td>
<td>continuous</td>
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<td>$\text{WKL}_0$</td>
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<tr>
<td></td>
<td>l.s.c.</td>
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**FIN**