Caristi's fixed point theorem and strong systems of arithmetic

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Fixed point theorems in analysis

Theorem (Banach, 1922)

Let $\mathcal{X}$ be a complete metric space and $f\colon \mathcal{X} \to \mathcal{X}$ be a contraction; that is, there is $\rho < 1$ such that $d(f(x), f(y)) < \rho \cdot d(x, y)$ for all $x, y \in X$. Then, there is $x_* \in X$ such that $f(x_*) = x_*$.

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Theorem (Brouwer, 1910)

Let D be a disk in $\mathbb{R}^n$ and $f\colon D \to D$ be continuous. Then, there is $x_* \in X$ such that $f(x_*) = x_*$.

Caristi systems and Caristi's fixed point theorem

Definition

A **Caristi system** is a triple $(\mathcal{X}, V, f)$, where

- $\mathcal{X}$ is a complete separable metric space,
- $V : \mathcal{X} \to \mathbb{R}^{\geq 0}$ is a lower semi-continuous function, and
- $f : \mathcal{X} \to \mathcal{X}$ is an **arbitrary function**,

such that

$$\forall x \in \mathcal{X} \; \Big(d(x, f(x)) \leq V(x) - V(f(x)) \Big).$$

Theorem (Caristi, 1976)

If $(\mathcal{X}, V, f)$ is a Caristi system, then f has a fixed point.

- Henceforth, a metric space is a complete separable metric space.
- We call the V in a Caristi system a potential.

Proofs of Caristi's theorem

1. Caristi's proof (simplified by Chi Song Wong)

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2. Proof by EVP.

Theorem (Ekeland, 1974)

Every lower semi-continuous function $V : \mathcal{X} \to \mathbb{R}_{\geq 0}$ has a critical point, i.e. a point $x_* \in \mathcal{X}$ such that

$$\forall y \in \mathcal{X} \left(d(x_*, y) \leq V(x_*) - V(y) \to y = x_* \right)$$

The Big Five subsystems of second-order arithmetic

- Two sorts: natural numbers, sets of naturals.
- Real numbers, infinite trees, etc. can all be coded in SOA.
- All systems have Σ^0_1-induction, elementary arithmetical axioms.

RCA_0	Computable sets exist (Δ^0_1 comprehension)
WKL_0	RCA_0 + "every infinite binary tree has an infinite path"
ACA_0	$\{n \in \mathbb{N} : \varphi(n)\}$ exists, φ arithmetical
ATR_0	Transfinite recursion for arithmetical formulas.
Π^1_1-CA_0	$\{n \in \mathbb{N} : \forall X \subseteq \mathbb{N}\ \varphi(n, X)\}$ exists, φ arithmetical

The strength of continuous Caristi's theorem

Theorem (F-D S T Y)

The following are equivalent over RCA_0.

1. ACA_0.
2. *Caristi's theorem for continuous functions.*
3. *Caristi's theorem for continuous potentials and continuous functions.*

Proof. $(1 \to 2)$. By EVP.

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$(3 \to 1)$. Use the fact that ACA_0 is equivalent to the statement that every decreasing sequence of positive reals has an infimium.

Compactness and Caristi's theorem

Theorem (F-D S T Y)

For **compact** *metric spaces* $\mathcal{X}$*:*

- *Caristi for l.s.c.* V *and continuous* f *is equivalent to* WKL_0.
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Baire functions:

- Continuous functions are Baire class 0
- ω-limits of Baire class $< \xi$ functions are Baire class ξ.

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Theorem (F-D S T Y)

For **compact** *metric spaces* $\mathcal{X}$*:*

- *Caristi for l.s.c.* V *and Baire class 1* f *is equivalent to* ACA_0.

Caristi vs. ATR_0

Theorem (F-D S T Y)

Caristi for Baire class 1 *functions* $f\colon \mathcal{X} \to \mathcal{X}$ *(with arbitrary* $\mathcal{X}$ *and l.s.c.* V*) implies* ATR_0.

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Caristi for Baire class 1 functions $f\colon \mathcal{X} \to \mathcal{X}$ (with arbitrary $\mathcal{X}$ and l.s.c. V) implies ATR_0.

Facts:

1. $\varphi\text{-}TR \ := \ \forall{\prec}\Big(WO(\prec) \to \exists Z \ \forall\xi \ \forall n \ \big(n \in Z_\xi \leftrightarrow \varphi(n, Z_{\prec\xi})\big)\Big)$
   - $Z_\xi = \{m \in \mathbb{N} : \langle m, \xi\rangle \in Z\}$,
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2. $\mathsf{ATR}_0 \equiv \mathsf{RCA}_0 + \Sigma^0_1\text{-}TR$.
3. If φ is Σ^0_1, there is a sequence of trees $(T_\xi)_{\xi \in \mathbb{N}}$ such that any path g through T_ξ codes Z_ξ.

Caristi for Baire class 1 F implies ATR_0.

Proof idea. Fix $(T_i)_{i<\omega}$. We define a Caristi system $(\mathcal{X}, F, V)$ as follows.

- $\mathcal{X}$ be the set of sequences of paths $(g_i)_{i<\omega}$.
- $F(\vec{g}) = \lim_{n\to\omega} F_n$, where $F_n(\vec{g})$ replaces g_ξ by an 'n-approximation' of the path through T_ξ, whenever:
  1. $g_\xi(n) \notin T_\xi$, and
  2. $\xi < n$ is the $\prec$-minimum satisfying 1.
- $V(\vec{g}) = \sum\{2^{-n} : g_n \text{ is not a path through } T_i\}$.

This defines a Caristi system, whose fixed point $(g_i^*)_{i<\omega}$ such that each g_i^* is a path through T_i. □

Leftmost paths

Theorem (Marcone)

Π^1_1-CA_0 *is equivalent to the statement "every ill-founded tree* $T \subseteq \mathbb{N}^{<\mathbb{N}}$ *has a leftmost path."*

Definition (Towsner)

- The **transfinite leftmost path principle** states that if $T \subseteq \mathbb{N}^{<\mathbb{N}}$ is ill-founded and α is a well-order, then there is a path f^* through T such that no path through T is both $\Sigma_\alpha^{T \oplus f^*}$ and to the left of f^*.
- TLPP_0 is RCA_0 plus the transfinite leftmost path principle.

TLPP_0 is strictly between ATR_0 and Π^1_1-CA_0.

Caristi's fixed point theorem for Baire functions

Theorem (F-D S T Y)

Caristi for Baire functions $f\colon \mathcal{X} \to \mathcal{X}$ (with arbitrary $\mathcal{X}$ and l.s.c. V) is equivalent to TLPP_0.

Thus in the general case:

- Caristi is equivalent to TLPP_0
- Ekeland is equivalent to Π^1_1-CA_0

Strength of Caristi's proof

Recall: Caristi's proof relies on uncountable Caristi sequences.

Definition

Fix a Caristi system $(\mathcal{X}, V, f)$.

A Caristi sequence (from x_0) is a well-order $(L, \prec)$ and a sequence $(x_\ell : \ell \in L) \subseteq \mathcal{X}$ such that

$$\begin{cases} x_{\min L} = x_0 \\ x_{S(\ell)} = f(x_\ell) \\ x_\ell = \lim_{k<\ell} x_k \qquad (\ell \in \mathsf{Lim}) \end{cases}$$

Maximal sequences

Call a Caristi sequence proper if the x_ℓ's are all distinct.

Lemma (Maximal sequence principle)

Given a Caristi system $(\mathcal{X}, V, f)$ with f arithmetical and $x_0 \in \mathcal{X}$, there is a proper Caristi sequence with no strict, proper extensions.

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Proof of Caristi's theorem by MSP: Any maximal sequence must have a last element, which is a fixed point of f. □

The closed orbit principle

Closed orbit: If $(\mathcal{X}, f)$ is a dynamical system and $x \in \mathcal{X}$, the closed orbit of x is the least topologically closed, f-closed set $\mathcal{O}^*$ such that $x \in \mathcal{O}^*$.

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Theorem (Banach fixed point theorem 2.0)

If $\mathcal{X}$ is a metric space and $f : \mathcal{X} \to \mathcal{X}$ is a contraction, then for any $x \in \mathcal{X}$, the closed orbit of $\mathcal{X}$ has a unique fixed point.

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Lemma (Closed orbit principle)

For every Caristi system $(\mathcal{X}, V, f)$ with f arithmetical and every $x_0 \in \mathcal{X}$, there is a $\subseteq$-least closed set $\mathcal{O}^*$ such that $x_0 \in \mathcal{O}^*$ and $(\forall x \in \mathcal{X})(x \in \mathcal{O}^* \to f(x) \in \mathcal{O}^*)$.

Inflationary fixed points

Definition

The arithmetical inflationary fixed point scheme is the scheme stating that if $F\colon 2^{\mathbb{N}} \to 2^{\mathbb{N}}$ is arithmetical and

$$\forall X\ (X \subseteq F(X)),$$

then there is a w.o. $(L, \prec)$ with max and sets $(X_\alpha : \alpha \in L)$ such that

$$\begin{cases} X_{\min L} = \varnothing \\ X_{S(\alpha)} = F(X_\alpha); \\ X_\gamma = \bigcup_{\alpha \prec \gamma} X_\alpha & (\gamma \in \mathsf{Lim}) \\ F(X_{\max L}) = X_{\max L}. \end{cases}$$

Stronger than Π^1_1-CA_0: does not require $X \subseteq Y \to F(X) \subseteq F(Y)$

Last slide!

Theorem (F-D S T Y)

The arithmetical inflationary fixed point scheme
$\equiv$ *the maximal sequence principle*
$\equiv$ *the closed orbit principle*

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Theorem (F-D S T Y)

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<table>
<tr><th>$\mathcal{X}$</th><th>f</th><th>V</th><th>Caristi</th><th>Ekeland</th></tr>
<tr><td rowspan="3">compact</td><td rowspan="2">continuous</td><td>continuous</td><td rowspan="2">WKL_0</td><td>WKL_0</td></tr>
<tr><td rowspan="2">l.s.c.</td><td rowspan="2">ACA_0</td></tr>
<tr><td>Baire 1</td><td>ACA_0</td></tr>
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<tr><td>Baire</td><td>TLPP_0</td></tr>
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