Turing jumps again

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Tuesday 17-10-2017
Fourth International Wormshop, Moscow

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  6. Rules: MP and Necessitation for all modalities.
- Rich logic and wide range of applicability

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$$[\xi]^d_T\varphi \leftrightarrow \Box_T\varphi \vee \exists \psi \ \Big(\forall n \, \exists \, \zeta{<}\xi \ [\zeta]_T\psi(\dot{n}) \wedge \Box_T(\forall x \psi(x) \to \varphi) \Big).$$

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- **Lemma:** $[\alpha]^d_T\varphi \Leftrightarrow [\alpha+1]_T\varphi$.

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- However, does not tie up with the Turing jump hierarchy and proves too much consistency
- Friedman, Godlfarb and Harrington come to the rescue!

- Definition (Witness-comparison relation)

  For $\phi := \exists x\ \phi_0(x)$ and $\psi := \exists x\ \psi_0(x)$ we define

$$\begin{array}{lcl} \phi \leq \psi & := & \exists x\ (\phi_0(x) \wedge \forall\, y{<}x\ \neg\psi_0(x)) \quad \text{and,} \\ \phi < \psi & := & \exists x\ (\phi_0(x) \wedge \forall\, y{\leq}x\ \neg\psi_0(x)). \end{array}$$

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- Theorem (Rosser's Theorem)

  Let T be a consistent c.e. theory extending EA. There is some $\rho \in \Sigma^0_1$ which is undecidable in T. That is,

$$\begin{array}{ll} T \nvdash \rho & \textit{and,} \\ T \nvdash \neg\rho. & \end{array}$$

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  - **Proof** Consider $\rho \leftrightarrow \neg(\Box\rho < \Box\neg\rho)$.

Lemma

Let A and B be some formulas of logical complexity Σ^0_{n+1} for $n < \omega$.

1. $\mathrm{EA} \vdash A \wedge \neg B \to (A < B)$;
2. $\mathrm{EA} \vdash (A < B) \to (A \leq B)$;
3. $\mathrm{EA} \vdash (A \leq B) \to A$;
4. $\mathrm{EA} \vdash (A \leq B) \to \neg(B < A)$ *and consequently;*
5. $\mathrm{EA} \vdash (A < B) \to \neg(B \leq A)$;
6. $\mathrm{EA} \vdash [(B \leq B) \vee (A \leq A)] \ \to \ [(A \leq B) \vee (B < A)]$;
7. $\mathrm{EA} \vdash A \wedge \neg(A \leq B) \to B$.

- Theorem (FGH theorem)

  Let T be any computably enumerable theory extending EA. For each $\sigma \in \Sigma^0_1$ we have that there is some $\rho \in \Sigma^0_1$ so that

$$\mathrm{EA} \vdash \Diamond_T \top \to \left(\sigma \leftrightarrow \Box_T \rho\right).$$

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  - We wish to stretch this further

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- **proposition**
  Let T be a computable theory extending EA and let ϕ be a Σ^0_{n+1} formula. We have that

$$\mathrm{EA} \vdash \phi \to [n]^{\mathsf{True}}_T \phi.$$

Theorem

Let T be any computably enumerable theory extending EA and let $n < \omega$. For each $\sigma \in \Sigma^0_{n+1}$ we have that there is some $\rho_n \in \Sigma^0_{n+1}$ so that

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- Just using Σ^0_{n+1}-completeness now
- $\langle n+1 \rangle^{\mathrm{True}}_T \top$ implies $\mathrm{I}\Sigma_n$
- In turn, $\mathrm{I}\Sigma_n$ implies Σ_1-collection
- So that $\sigma \leq [n]^{\mathrm{True}}_T \rho_n$ is a Σ^0_{n+1} sentence

Corollary

Let T be a c.e. theory extending EA and let $n \in \mathbb{N}$. For each formulas φ, ψ there is some σ so that

$$T \vdash ([n]^{\mathsf{True}}_T \varphi \ \vee [n]^{\mathsf{True}}_T \psi) \ \leftrightarrow \ [n]^{\mathsf{True}}_T \sigma.$$

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  4. *A is definable on the standard model by a formula of the form $[n]^{\mathrm{True}}_T \rho(\dot{x})$;*
  5. *A is definable on the standard model by a formula of the form $[n]^{\mathrm{True}}_T \rho(\dot{x})$ where $\rho(x) \in \Sigma^0_{n+1}$;*

- $[n+1]_T^{\text{Omega}} \varphi := \exists \psi \left(\forall x\, [n]_T^{\text{Omega}} \psi(\dot{x}) \wedge \Box_T (\forall x\, \psi(x) \to \varphi) \right)$
- $[n]_T^{\text{Omega}}$ is a Σ^0_{2n+1}-formula.

- Lemma

  Let T be a computable theory extending EA *and let* ϕ *be a* Σ^0_{2n+1} *formula. We have that*

$$\mathrm{EA} \vdash \phi \to [n]_T^{\text{Omega}} \phi.$$

Proof.

By an external induction on n where each inductive step requires the application of an additional omega-rule. □

Corollary

Let T be any computably enumerable theory extending EA and let $n < \omega$. For each $\sigma \in \Sigma^0_{2n+1}$ we have that there is some $\rho_n \in \Sigma^0_{2n+1}$ so that

$$\mathrm{EA} \vdash \langle n \rangle^{\mathsf{Omega}}_T \top \to \Big(\sigma \leftrightarrow [n]^{\mathsf{Omega}}_T \rho_n\Big).$$

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  - Runs out of phase!
  - We wish to use the best of both worlds

- $$\begin{aligned} [0]^{\Box}_{T}\phi &:= \Box_T \phi, \quad \text{and} \\ [n+1]^{\Box}_{T}\phi &:= \Box_T \phi \ \vee \ \exists \psi \bigvee_{0 \leq m \leq n} \Big(\langle m \rangle^{\Box}_{T} \psi \ \wedge \ \Box(\langle m \rangle^{\Box}_{T} \psi \to \phi) \Big). \end{aligned}$$

proposition

Let T be a sound c.e. theory extending EA. We have for all $n \in \mathbb{N}$ that

1. $\mathrm{EA} \vdash \ \forall \varphi \ ([n]^{\Box}_{T}\varphi \ \to \ [n]^{\mathrm{True}}_{T}\varphi)$;
2. $\mathrm{EA} \vdash \ \langle n \rangle^{\mathrm{True}}_{T}\top \ \to \ \forall \varphi \ ([n+1]^{\Box}_{T}\varphi \leftrightarrow [n+1]^{\mathrm{True}}_{T}\varphi)$;
3. $\mathrm{EA} \vdash \ [n]^{\mathrm{True}}_{T}\Big(\ \forall \varphi \ ([n]^{\Box}_{T}\varphi \leftrightarrow [n]^{\mathrm{True}}_{T}\varphi) \ \Big)$;
4. $\mathbb{N} \models \ \forall \varphi \ ([n]^{\Box}_{T}\varphi \leftrightarrow [n]^{\mathrm{True}}_{T}\varphi)$.

Theorem

Let T be a c.e. theory. We have for all $A \subseteq \mathbb{N}$ that the following are equivalent

1. *A is c.e. in $\emptyset^{(n)}$;*
2. *A is 1-1 reducible to $\emptyset^{(n+1)}$;*
3. *A is definable on the standard model by a formula of the form $[n]^{\Box}_{T}\rho(\dot{x})$;*

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$$[\zeta]\phi \quad :\Leftrightarrow \quad \Box\phi \ \vee \ \exists\psi\,\exists\,\xi{<}\zeta\ (\langle\xi\rangle\psi \ \wedge \ \Box(\langle\xi\rangle\psi \to \phi)). \qquad (1)$$

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- **Observation:**
  $$\mathbb{N} \models \langle\xi\rangle\top$$
  for any $\xi < \Lambda$

Lemma

Let $\xi < \zeta$ be ordinals and T a c.e. theory extending ECA_0. We have

1. $\mathrm{ECA}_0 \vdash \forall \varphi \ \big(\langle \xi \rangle^{\Box}_{T} \varphi \ \to \ [\zeta]^{\Box}_{T} \langle \xi \rangle^{\Box}_{T} \varphi \big)$;
2. $\mathrm{ECA}_0 \vdash \forall \varphi \ \big([\xi]^{\Box}_{T} \varphi \ \to \ [\zeta]^{\Box}_{T} \varphi \big)$.

Proof.

Almost directly from the defining recursion □

Here, ECA_0 is the second order pendant of *Elementary Arithmetic*

Löb's axioms will follow from distribution and the 4 axiom:

Lemma

Extend GL *with a new operator* $\blacksquare$ *and the following axioms for all formulas* ϕ, *and* ψ*:*

1. $\vdash \Box\phi \to \blacksquare\phi$,
2. $\vdash \blacksquare(\phi \to \psi) \to (\blacksquare\phi \to \blacksquare\psi)$ *and,*
3. $\vdash \blacksquare\phi \to \blacksquare\blacksquare\phi$,

and call the resulting system $\mathsf{GL}^{\blacksquare}$*.*
Then for all ϕ,

$$\mathsf{GL}^{\blacksquare} \vdash \blacksquare(\blacksquare\phi \to \phi) \to \blacksquare\phi.$$

Theorem

Let T be some c.e. theory extending ECA_0.

- ECA_0 + *?? proves that the $[\alpha]^{\Box}_{T}$ predicate is closed under both conjunctions and disjunctions;*
- ECA_0 + *?? proves that all the rules and axioms of GLP are sound wr.t. T by interpreting $[\alpha]$ as $[\alpha]^{\Box}_{T}$.*

Proof.

Using the cross-axioms, simulataneously prove:

1. Closure under conjunctions: $\forall\varphi\,\forall\psi\,\left([\alpha]^{\Box}_{T}\varphi\wedge[\alpha]^{\Box}_{T}\psi\ \ \leftrightarrow[\alpha]^{\Box}_{T}(\varphi\wedge\psi)\right)$;
2. Distributivity: $\forall\varphi,\psi\,\left([\alpha]^{\Box}_{T}(\varphi\to\psi)\to([\alpha]^{\Box}_{T}\varphi\to[\alpha]^{\Box}_{T}\psi)\right)$;
3. Closure under disjunctions: $\forall\varphi\,\forall\psi\,\exists\chi\,\left([\alpha]^{\Box}_{T}\varphi\vee[\alpha]^{\Box}_{T}\psi\ \ \leftrightarrow[\alpha]^{\Box}_{T}\chi\right)$;
4. Transitivity: $\forall\varphi\,\left([\alpha]^{\Box}_{T}\varphi\to[\alpha]^{\Box}_{T}[\alpha]^{\Box}_{T}\varphi\right)$.

□

Theorem

The logic GLP_Λ *is sound for strong enough theories* T *under the interpretation* $\Box \mapsto [\lambda]_T^{\Box,\Lambda}$.

- **Theorem** Let T be any c.e. theory, let $\xi < \Lambda$ for a natural ordinal notation system, and let $A \subseteq \mathbb{N}$. The following are equivalent

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- No longer runs out of phase

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- Iterated Provability Classes:

$$\langle \xi, \varphi \rangle \in X \quad \leftrightarrow \quad \Big(\Box_T \varphi \vee \exists \psi \, \exists \zeta {<} \xi \, \big(\forall n \, \langle \zeta, \psi(\overline{n}) \rangle \in X \wedge \Box_T (\forall x \psi(x) \to \varphi) \big) \Big)$$

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- Iterated Turing Jump Classes:

$$\langle \xi, \varphi \rangle \in X \quad \leftrightarrow \quad \Big(\Box_T \varphi \vee \exists \psi \, \exists \zeta {<} \xi \, \big(\langle \zeta, \neg \psi \rangle \notin X \wedge \Box_{T|X+??} (\langle \zeta, \neg \psi \rangle \notin X \to \varphi) \big) \Big)$$

Given a formula $\pi(c, \lambda, \phi)$, we introduce the notation $[c : \lambda]_\pi \phi = \pi(c, \lambda, \phi)$, as well as $[\lambda]_\pi \phi = \exists c [c : \xi]_\pi \phi$.
A Λ-*uniform proof predicate over* T is a formula $\pi(c, \lambda, \phi)$ (with all free variables shown) satisfying

1. $T \vdash \mathrm{I}\Sigma^0_1(\pi)$;
2. $T \vdash \forall \lambda \forall \phi \ (\Box_T \phi \to [\lambda]_\pi \phi)$;
3. $T \vdash \forall \lambda \forall \phi \forall \psi \ \big([\lambda]_\pi (\psi \to \phi) \wedge [\lambda]_\pi \psi \to [\lambda]_\pi \phi\big)$;
4. $T \vdash \forall c \, \forall \xi {\leq_\Lambda} \lambda \, \forall \phi \ \big([c : \xi]_\pi \phi \to [c : \lambda]_\pi \phi\big)$;
5. $T \vdash \forall c \, \forall \lambda \, \forall \phi \ \big([c : \lambda]_\pi \phi \to [\lambda]_\pi [\dot{c} : \dot{\lambda}]_\pi \dot{\phi}\big)$;
6. $T \vdash \forall c \forall \lambda \, \forall \phi \ \big(\langle c : \lambda \rangle_\pi \phi \to [\lambda]_\pi \langle \dot{c} : \dot{\lambda} \rangle_\pi \dot{\phi}\big)$;
7. $T \vdash \forall \lambda \, \forall \, \xi {<_\Lambda} \lambda \, \forall \phi \ \big(\langle \xi \rangle_\pi \phi \to [\lambda]_\pi \langle \dot{\xi} \rangle_\pi \dot{\phi}\big)$.

We have arithmetical completeness for such predicates

Complexity classes anew:

Definition

Let T be a c.e. theory. We define

- $\Delta_0^\Box := \Sigma_0^\Box := \Pi_0^\Box := \Delta_0^0$;
- $\Sigma_{\alpha+1}^\Box = \Sigma_\alpha^\Box \cup \Pi_\alpha^\Box \cup \{[\alpha]_T^\Box \varphi(\dot{x}) \mid \varphi(x) \in \text{Form}\}$ for $\alpha > 0$;
- $\Pi_{\alpha+1}^\Box = \Sigma_\alpha^\Box \cup \Pi_\alpha^\Box \cup \{\langle\alpha\rangle_T^\Box \varphi(\dot{x}) \mid \varphi(x) \in \text{Form}\}$ for $\alpha > 0$;
- $\Sigma_\lambda^\Box := \Pi_\lambda^\Box := \bigcup_{\alpha<\lambda} \Sigma_\alpha^\Box$ for $\lambda \in \text{Lim}$.

Definition

Let Γ be a class of formulas. For ordinals $\alpha, \beta < \Lambda$ and T a c.e. theory we define $\beta{-}\mathtt{RFN}^{\Lambda}_{T}(\Gamma)$ to be the schema $[\beta]^{\Box}_{T}\varphi \to \varphi$ for $\varphi \in \Gamma$.

Instead of writing $0{-}\mathtt{RFN}^{\Lambda}_{T}(\Gamma)$ we shall just write $\mathtt{RFN}^{\Lambda}_{T}(\Gamma)$.

We can now easily state and prove various equivalences between consistency statements and reflection principles.

Theorem

Let T be a c.e. theory containing $\mathtt{ECA}_0$.

1. $\mathtt{ECA}_0 \vdash \ \mathtt{RFN}^{\Lambda}_{T}(\Pi^{\Box}_{\alpha+1}) \ \equiv \ \langle\alpha\rangle^{\Box}_{T}\top$;
2. For $\beta \leq \alpha$, we have $\mathtt{ECA}_0 \vdash \ \beta{-}\mathtt{RFN}^{\Lambda}_{T}(\Pi^{\Box}_{\alpha+1}) \ \equiv \ \langle\alpha\rangle^{\Box}_{T}\top$;
3. For $\beta > \alpha$ we have that $\mathtt{ECA}_0 \vdash \ \beta{-}\mathtt{RFN}^{\Lambda}_{T}(\Pi^{\Box}_{\alpha+1}) \equiv \langle\beta\rangle^{\Box}_{T}\top$;
4. For $\beta > \alpha$ we have that
   $\mathtt{ECA}_0 \vdash \ \beta{-}\mathtt{RFN}^{\Lambda}_{T}(\Pi^{\Box}_{\alpha+1}) \equiv \langle\max\{\alpha,\beta\}\rangle^{\Box}_{T}\top$.

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$$\langle n{+}1\rangle\varphi \ \equiv_n \ \{\langle n\rangle\varphi,\ \langle n\rangle(\varphi\wedge\langle n\rangle\varphi),\ \langle n\rangle(\varphi\wedge\langle n\rangle(\varphi\wedge\langle n\rangle\varphi)),\ldots\}$$

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- Here, $\Gamma \equiv_n \Delta$ means $\Gamma \vdash \langle n\rangle\psi \iff \Gamma \vdash \langle n\rangle\psi$

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- Given a worm A and formula φ, then $Q(A,\varphi)$ is defined as follows:
  - $Q(\top,\varphi) \ := \ \top;$
  - $Q(\langle\xi\rangle A',\varphi) \ := \ \langle\xi\rangle(\varphi \wedge Q(A',\varphi)).$
- '**Theorem**' Let $\zeta < \xi < \Gamma$ and let the set of worms $\langle\zeta\rangle\Theta$ be $<$-cofinal in $\langle\xi\rangle\top$, then over GLP_Γ we have that

$$\langle\xi\rangle\varphi \ \equiv_\zeta \ Q(\langle\zeta\rangle\Theta,\varphi)$$

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- Generalization (Hermo-Reyes, JjJ): for each $\xi, \zeta, \theta < \Gamma$, there is a worm A so that over Turing-Schmerl Calculus we have

$$(\mathrm{EA}^+)^\xi_\zeta \equiv_\theta \mathrm{EA}^+ + A$$

THANK YOU!