Iteration in Residuated Structures

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Standard example: the algebra of languages over an alphabet, possibly with the empty word.

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Standard example: the algebra of languages without the empty word.

Multiplicative-Only Fragment (the Lambek Calculus with Iteration, $\mathbf{L}^{+}_{\omega}$)

(for the $*$-continuous case;
cf. $\mathbf{ACT}_{\omega}$ by Buszkowski and Palka 2005–08)

$$A \to A$$

$$\frac{A\,\Pi \to B}{\Pi \to A \setminus B}\ \text{, where } \Pi \text{ is not empty} \qquad \frac{\Pi \to A \quad \Gamma\, B\, \Delta \to C}{\Gamma\, \Pi\, (A \setminus B)\, \Delta \to C}$$

$$\frac{\Pi\, A \to B}{\Pi \to B \,/\, A}\ \text{, where } \Pi \text{ is not empty} \qquad \frac{\Pi \to A \quad \Gamma\, B\, \Delta \to C}{\Gamma\, (B \,/\, A)\, \Pi\, \Delta \to C}$$

$$\frac{\Gamma \to A \quad \Delta \to B}{\Gamma, \Delta \to A \cdot B} \qquad \frac{\Gamma, A, B, \Delta \to C}{\Gamma, A \cdot B, \Delta \to C}$$

$$\frac{\Gamma_1 \to A \quad \ldots \quad \Gamma_n \to A}{\Gamma_1, \ldots, \Gamma_n \to A^{+}}\ (n \geq 1) \qquad \frac{\Gamma, A^n, \Delta \to C \text{ for all } n \geq 1}{\Gamma, A^{+}, \Delta \to C}$$

$$\frac{\Pi \to A \quad \Gamma\, A\, \Delta \to C}{\Gamma\, \Pi\, \Delta \to C}\ (\text{cut})$$

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Proof idea: following Buszkowski & Palka for $\mathbf{ACT}_\omega$, encode the totality problem for context-free grammars. The key trick that allows avoiding $\vee$ and $\wedge$ is the usage of *Lambek grammars with unique type assignment* [Safiullin 2007].

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CFG $\to$ Lambek categorial grammar.

$a_1 \triangleright A_1$, $a_2 \triangleright A_2$, C is the goal category. (Alphabet $\{a_1, a_2\}$)

$a_1 ... a_n \in \mathcal{L} \iff A_1 \dots A_n \to C$ is derivable.

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Open question: Safiullin's result is not known for the case with empty word. Therefore, we cannot yet replace $^+$ with *.

On The Other Side...

Pratt's axiomatisation for general (non necessarily $*$-continuous) action algebras (a variant with positive iteration):

$$A \to A \qquad (A \cdot B) \cdot C \to A \cdot (B \cdot C) \qquad A \cdot (B \cdot C) \to (A \cdot B) \cdot C$$

$$\frac{A \to C \,/\, B}{A \cdot B \to C} \qquad \frac{A \cdot B \to C}{A \to C \,/\, B} \qquad \frac{B \to A \setminus C}{A \cdot B \to C} \qquad \frac{A \cdot B \to C}{B \to A \setminus C}$$

$$\frac{A \to B \quad B \to C}{A \to C} \qquad \frac{A \to B_i}{A \to B_1 \vee B_2} \qquad \frac{A_1 \to B \quad A_2 \to B}{A_1 \vee A_2 \to B}$$

$$\frac{A_i \to B}{A_1 \wedge A_2 \to B} \qquad \frac{A \to B_1 \quad A \to B_2}{A \to B_1 \wedge B_2}$$

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NB: Pratt 1990 doesn't cite Lambek 1958 (but cites Girard 1987).

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Open question 2: lower complexity bounds for $\mathbf{ACT}_{\mathrm{Pratt}}$ without $\vee$ and $\wedge$.

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- **Work in progress:** cut elimination in $\mathbf{L}^+_\infty$, cf. Savateev's talk today.

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We allow to use the conclusion of the negative rule to be used as a premise in its derivation tree (*backlink*).

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Open question: a cut-free system? (cf. Jipsen 2004 for a different approach).

Language Interpretation

$$w(A) \subseteq \Sigma^+$$

$$w(A \setminus B) = w(A) \setminus w(B) = \{u \in \Sigma^+ \mid (\forall v \in w(A))\ vu \in w(B)\}$$
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Theorem (M. Pentus 1995)

$\mathrm{L} \vdash A \to B \iff w(A) \subseteq w(B)$ *for all* w.

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Open question: completeness of $\mathbf{L}_\omega^+$.

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A partial result [Ryzhkova 2013]: completeness for the fragment without $\cdot$, where $^+$ appears only as $A^+ \setminus B$ or $B \,/\, A^+$.

Even More Complexity: Iteration and the Exponential

The exponential, !, governed by the following rules:

$$\frac{\Gamma, A, \Delta \to C}{\Gamma, !A, \Delta \to C} \qquad \frac{!A_1, \ldots, !A_n \to B}{!A_1, \ldots, !A_n \to !B} \qquad \frac{\Gamma, \Delta \to C}{\Gamma, !A, \Delta \to C}$$

$$\frac{\Gamma, \Phi, !A, \Delta \to C}{\Gamma, !A, \Phi, \Delta \to C} \qquad \frac{\Gamma, !A, \Phi, \Delta \to C}{\Gamma, \Phi, !A, \Delta \to C} \qquad \frac{\Gamma, !A, !A, \Delta \to C}{\Gamma, !A, \Delta \to C}$$

allows encoding *derivation from a finite theory* as a derivation of one formula:

$$A_1 \to B_1, \ldots, A_k \to B_k \vdash \Gamma \to C \iff !(A_1 \setminus B_1), \ldots, !(A_k \setminus B_k), \Gamma \to C$$

Even More Complexity: Iteration and the Exponential

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$$\frac{\Gamma, A, \Delta \to C}{\Gamma, !A, \Delta \to C} \qquad \frac{!A_1, \ldots, !A_n \to B}{!A_1, \ldots, !A_n \to !B} \qquad \frac{\Gamma, \Delta \to C}{\Gamma, !A, \Delta \to C}$$

$$\frac{\Gamma, \Phi, !A, \Delta \to C}{\Gamma, !A, \Phi, \Delta \to C} \qquad \frac{\Gamma, !A, \Phi, \Delta \to C}{\Gamma, \Phi, !A, \Delta \to C} \qquad \frac{\Gamma, !A, !A, \Delta \to C}{\Gamma, !A, \Delta \to C}$$

allows encoding *derivation from a finite theory* as a derivation of one formula:

$$A_1 \to B_1, \ldots, A_k \to B_k \vdash \Gamma \to C \iff !(A_1 \setminus B_1), \ldots, !(A_k \setminus B_k), \Gamma \to C$$

Therefore,

- **L** with ! is undecidable (Σ^0_1-complete): encoding derivations in semi-Thue systems (actually a subset of rules for ! is sufficient, see Scedrov's talk today);

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Therefore,

- **L** with ! is undecidable (Σ^0_1-complete): encoding derivations in semi-Thue systems (actually a subset of rules for ! is sufficient, see Scedrov's talk today);
- **L** with ! and * is Π^1_1-hard: encoding Kozen 2002 (deriving Horn clauses in *-continuous Kleene algebra is Π^1_1-complete).

Open question: Π^1_1 upper bound.

Thanks$^+$