Iteration in Residuated Structures

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Residuated Kleene Algebras (Action Algebras)

\[ \langle \mathcal{A}; \cdot, 1, \backslash, /, \lor, \land, *, \leq \rangle \]
Residuated Kleene Algebras (Action Algebras)

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- $\cdot$ gives a monoid structure, $1$ is the unit;
Residuated Kleene Algebras (Action Algebras)

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- \cdot gives a monoid structure, 1 is the unit;
- \setminus and / are residuals of \cdot:

\[
\quad a \leq c/b \iff a \cdot b \leq c \iff b \leq a\setminus c
\]
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- \lor and \land are for the lattice structure:
  \[ a \lor b = \sup\{a, b\}, \quad a \land b = \inf\{a, b\}. \]
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- \( \lor \) and \( \land \) are for the lattice structure:
  \[ a \lor b = \sup \{a, b\}, \quad a \land b = \inf \{a, b\}. \]
- \( * \), the general case: \( 1 \lor a \lor (a^* \cdot a^*) \leq a^* \), and if \( 1 \lor a \lor (b \cdot b) \leq b \), then \( a^* \leq b \).
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- * , the general case: \[ 1 \lor a \lor (a^* \cdot a^*) \leq a^*, \text{ and if } \]
  \[ 1 \lor a \lor (b \cdot b) \leq b, \text{ then } a^* \leq b. \]
- * , the *-continuous case: \[ p \cdot q^* \cdot r = \sup \{p \cdot q^n \cdot r \mid n \geq 0\} \]
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- *, the *-continuous case: \[ p \cdot q^* \cdot r = \sup \{p \cdot q^n \cdot r \mid n \geq 0\} \]


**Standard example:** the algebra of languages over an alphabet, possibly with the empty word.
In This Talk...

... we consider the *positive* version of Kleene iteration \((+ \text{ instead of } \ast)\):

\[ a \lor (a + (a + a)) \leq a + a, \text{ and if } a \lor (b \cdot b) \leq b, \text{ then } a + a \leq b \text{ (for the } \lor \text{-continuous case)}. \]

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- a semigroup instead of a monoid;
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- $p \cdot q^+ \cdot r = \sup\{p \cdot q^n \cdot r \mid n \geq 1\}$ (for the $^*$-continuous case).
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**Standard example:** the algebra of languages without the empty word.
Multiplicative-Only Fragment (the Lambek Calculus with Iteration, $L^+_\omega$)

(for the $\ast$-continuous case; cf. $\text{ACT}_\omega$ by Buszkowski and Palka 2005–08)

\[
\begin{align*}
A & \rightarrow A \\
A \Pi \rightarrow B & \quad \Pi \rightarrow A \setminus B, \text{ where } \Pi \text{ is not empty} \\
\Pi A \rightarrow B & \quad \Pi \rightarrow B / A, \text{ where } \Pi \text{ is not empty} \\
\Gamma \rightarrow A & \quad \Delta \rightarrow B \\
\Gamma, \Delta \rightarrow A \cdot B & \\
\Gamma_1 \rightarrow A & \quad \ldots \quad \Gamma_n \rightarrow A \\
\Gamma_1, \ldots, \Gamma_n \rightarrow A^+ & \quad (n \geq 1) \\
\Pi \rightarrow A & \quad \Gamma A \Delta \rightarrow C \\
\Gamma \Pi \Delta \rightarrow C & \quad \text{(cut)} \\
\Gamma, A, B, \Delta \rightarrow C & \\
\Gamma, A \cdot B, \Delta \rightarrow C & \\
\Gamma, A^n, \Delta \rightarrow C & \quad \text{for all } n \geq 1 \\
\Gamma, A^+, \Delta \rightarrow C
\end{align*}
\]
Complexity Result

Theorem
\( L_\omega^+ \) is \( \Pi^0_1 \)-complete.

Proof idea: following Buszkowski & Palka for \( ACT_\omega \), encode the totality problem for context-free grammars. The key trick that allows avoiding \( \lor \) and \( \land \) is the usage of Lambek grammars with unique type assignment [Safiullin 2007].

CFG \( \rightarrow \) Lambek categorial grammar.

\[ a_1 \triangleleft A_1, a_2 \triangleleft A_2, C \text{ is the goal category. (Alphabet } \{ a_1, a_2 \} \text{)} \]

\( a_1 \ldots a_n \in L \iff A_1 \ldots A_n \rightarrow C \text{ is derivable.} \)

Checking derivability of \( (A + \cdot B + \cdot C) + \cdot \rightarrow C \) is roughly equivalent to checking totality for the CFG.

Open question: Safiullin's result is not known for the case with empty word. Therefore, we cannot yet replace \( + \) with \( * \).
Theorem

$L^+_\omega$ is $\Pi^0_1$-complete.

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$\text{CFG} \rightarrow \text{Lambek categorial grammar.}$

$a_1 \triangleright A_1, \ a_2 \triangleright A_2, \text{ C is the goal category. (Alphabet } \{a_1, a_2\})$

$a_1...a_n \in \mathcal{L} \iff A_1 \ldots A_n \rightarrow C \text{ is derivable.}$

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**Theorem**

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Checking derivability of $(A^+ \cdot B^+)^+ \rightarrow C$ is roughly equivalent to checking totality for the CFG.

**Open question:** Safiullin’s result is not known for the case with empty word. Therefore, we cannot yet replace $^+$ with $^*$. 
Pratt’s axiomatisation for general (non necessarily $\ast$-continuous) action algebras (a variant with positive iteration):

\[
A \rightarrow A \quad (A \cdot B) \cdot C \rightarrow A \cdot (B \cdot C) \quad A \cdot (B \cdot C) \rightarrow (A \cdot B) \cdot C
\]

\[
\begin{align*}
A \rightarrow C / B & \quad A \cdot B \rightarrow C & \quad B \rightarrow A \setminus C & \quad A \cdot B \rightarrow C \\
A \cdot B \rightarrow C & \quad A \rightarrow C / B & \quad A \cdot B \rightarrow C & \quad B \rightarrow A \setminus C \\
A \rightarrow B & \quad B \rightarrow C & \quad A \rightarrow B_i & \quad A_1 \rightarrow B \quad A_2 \rightarrow B \\
A \rightarrow B & \quad B \rightarrow C & \quad A \rightarrow B_1 \lor B_2 & \quad A_1 \lor A_2 \rightarrow B \\
A_i \rightarrow B & \quad A \rightarrow B_1 & \quad A \rightarrow B_2 & \quad A \rightarrow B_1 \land B_2 \\
A_1 \land A_2 \rightarrow B & \quad A \rightarrow B \quad A \rightarrow B_1 \land B_2 & \quad \quad A \lor (B \cdot B) \rightarrow B \\
A \lor (A^+ \cdot A^+) \rightarrow A^+ & \quad A^+ \rightarrow B
\end{align*}
\]
On The Other Side...

Pratt’s axiomatisation for general (non necessarily $\ast$-continuous) action algebras (a variant with positive iteration):

$$
A \to A \quad (A \cdot B) \cdot C \to A \cdot (B \cdot C) \quad A \cdot (B \cdot C) \to (A \cdot B) \cdot C
$$

$$
\begin{align*}
A & \to C / B \\
A \cdot B & \to C \\
A & \to C / B \\
A \cdot B & \to C \\
B & \to A \setminus C \\
A \cdot B & \to C \\
B & \to A \setminus C
\end{align*}
$$

$$
\begin{align*}
A & \to B \quad B \to C \\
A & \to C
\end{align*}
$$

$$
\begin{align*}
A \to B_i \\
A & \to B_1 \lor B_2 \\
A_1 \lor A_2 & \to B \\
A_i & \to B \\
A_1 \land A_2 & \to B \\
A & \to B_1 \\
A & \to B_2 \\
A & \to B_1 \land B_2
\end{align*}
$$

$$
A \lor (A^+ \cdot A^+) \to A^+ \\
A \lor (B \cdot B) \to B \\
A^+ \to B
$$

NB: Pratt 1990 doesn’t cite Lambek 1958 (but cites Girard 1987).
Induction vs. *-continuity

\( \text{ACT}_\omega \) is \( \Pi^0_1 \)-complete (Buszkowski & Palka);
\( \text{ACT}_{\text{Pratt}} \) is in \( \Sigma^0_1 \) (r.e.)
Induction vs. *-continuity

\( \text{ACT}_\omega \) is \( \Pi^0_1 \)-complete (Buszkowski & Palka);
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Therefore:

▶ there exists an action algebra that is not *-continuous;
▶ the equational theories of all action algebras and *-continuous action algebras differ, even in the fragment without \( \lor \) and \( \land \) (for positive iteration).

Note that, as shown by Kozen, for the case without \( \backslash \) and \( / \) (but with \( \lor \)) the equational theories coincide.

Open question 1: construct a concrete example of a formula valid in all *-continuous action algebras, but not in all action algebras.

Open question 2: lower complexity bounds for \( \text{ACT}_{\text{Pratt}} \) without \( \lor \) and \( \land \).
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Therefore:

- there exists an action algebra that is not *-continuous;
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**Open question 1:** construct a concrete example of a formula valid in all *-continuous action algebras, but not in all action algebras.
Induction vs. ∗-continuity

$\text{ACT}_\omega$ is $\Pi^0_1$-complete (Buszkowski & Palka);

$\text{ACT}_{\text{Pratt}}$ is in $\Sigma^0_1$ (r.e.)

Therefore:

- there exists an action algebra that is not ∗-continuous;
- the equational theories of all action algebras and ∗-continuous action algebras differ, even in the fragment without $\lor$ and $\land$ (for positive iteration).

Note that, as shown by Kozen, for the case without $\backslash$ and $/$ (but with $\lor$) the equational theories coincide.

Open question 1: construct a concrete example of a formula valid in all ∗-continuous action algebras, but not in all action algebras.

Open question 2: lower complexity bounds for $\text{ACT}_{\text{Pratt}}$ without $\lor$ and $\land$. 
System with Non-well-founded Derivations ($L^+_{\infty}$)

\[
\begin{align*}
\frac{\Pi \rightarrow A}{\Pi \rightarrow A^+} & \quad \frac{\Pi_1 \rightarrow A \quad \Pi_2 \rightarrow A^+}{\Pi_1, \Pi_2 \rightarrow A^+} & \frac{\Gamma, A, \Delta \rightarrow C}{\Gamma, A^+, \Delta \rightarrow C} \quad \frac{\Gamma, A, A^+, \Delta \rightarrow C}{\Gamma, A^+, \Delta \rightarrow C}
\end{align*}
\]

We allow infinite branches in the proof tree.

Correctness condition: for every infinite branch there exists an active occurrence $A^+$ that undergoes the left rule infinitely many times.

Incorrect derivation example:

\[
p \rightarrow p
\]

\[
p \rightarrow p \quad p \rightarrow p^+ \rightarrow p^+ \rightarrow \ldots
\]

\[
p \rightarrow p \quad p \rightarrow p^+ \rightarrow \ldots
\]

\[
(\text{cut})
\]

$L^+_{\infty}$ is equivalent to $L^+_{\omega}$.

Work in progress: cut elimination in $L^+_{\infty}$, cf. Savateev's talk today.
System with Non-well-founded Derivations ($L^+_\omega$)

We allow infinite branches in the proof tree.
System with Non-well-founded Derivations ($L^+_\infty$)

\[
\frac{\Pi \rightarrow A}{\Pi \rightarrow A^+} \quad \frac{\Pi_1 \rightarrow A \quad \Pi_2 \rightarrow A^+}{\Pi_1, \Pi_2 \rightarrow A^+} \quad \frac{\Gamma, A, \Delta \rightarrow C}{\Gamma, A^+, \Delta \rightarrow C}
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\[
\begin{align*}
\Pi & \rightarrow A \\
\Pi & \rightarrow A^+ \\
\Pi_1 & \rightarrow A \\
\Pi_2 & \rightarrow A^+ \\
\Pi_1, \Pi_2 & \rightarrow A^+ \\
\Gamma, A, \Delta & \rightarrow C \\
\Gamma, A^+, \Delta & \rightarrow C \\
\Gamma, A^+, \Delta & \rightarrow C
\end{align*}
\]

- We allow infinite branches in the proof tree.
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\[
\begin{align*}
p & \rightarrow p \\
p^+ & \rightarrow p^+ \\
p^+ & \rightarrow p^+ \\
p^+ & \rightarrow p^+ \\
p^+ & \rightarrow p^+ \\
p^+ & \rightarrow p^+ \\
p & \rightarrow p \\
p & \rightarrow p \\
p^+ & \rightarrow p \\
p^+ & \rightarrow p \\
p & \rightarrow p \\
p^+ & \rightarrow p
\end{align*}
\]

\[
\begin{align*}
p, p^+ & \rightarrow p^+ \\
p^+ & \rightarrow p^+ \\
p & \rightarrow p
\end{align*}
\]

\[
\cdot \cdot \cdot
\]

\[
\begin{align*}
p & \rightarrow p \\
p & \rightarrow p \\
p^+ & \rightarrow p \\
p^+ & \rightarrow p
\end{align*}
\]

\[
\begin{align*}
p^+ & \rightarrow p \\
p^+ & \rightarrow p
\end{align*}
\]

\[
\text{(cut)}
\]

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System with Non-well-founded Derivations ($\mathbf{L}^+_\infty$)

\[
\begin{align*}
\frac{\Pi \rightarrow A}{\Pi \rightarrow A^+} & \quad \frac{\Pi_1 \rightarrow A \quad \Pi_2 \rightarrow A^+}{\Pi_1, \Pi_2 \rightarrow A^+} & \quad \frac{\Gamma, A, \Delta \rightarrow C \quad \Gamma, A, A^+, \Delta \rightarrow C}{\Gamma, A^+, \Delta \rightarrow C}
\end{align*}
\]

- We allow infinite branches in the proof tree.
- Correctness condition: for every infinite branch there exists an active occurrence $A^+$ that undergoes the left rule infinitely many times. Incorrect derivation example:

\[
\begin{align*}
p \rightarrow p & \quad p^+ \rightarrow p^+ \quad \therefore \\
p, p^+ \rightarrow p^+ & \quad p^+ \rightarrow p \\
p \rightarrow p & \quad p, p^+ \rightarrow p \\
p^+ \rightarrow p
\end{align*}
\]

- $\mathbf{L}^+_\infty$ is equivalent to $\mathbf{L}^+_\omega$.  

- Work in progress: cut elimination in $\mathbf{L}^+\infty$, cf. Savateev's talk today.
System with Non-well-founded Derivations ($L_\infty^+$)

\[
\frac{\Pi \to A}{\Pi \to A^+} \quad \frac{\Pi_1 \to A \quad \Pi_2 \to A^+}{\Pi_1, \Pi_2 \to A^+} \quad \frac{\Gamma, A, \Delta \to C}{\Gamma, A^+, \Delta \to C}
\]

- We allow infinite branches in the proof tree.
- Correctness condition: for every infinite branch there exists an active occurrence $A^+$ that undergoes the left rule infinitely many times. Incorrect derivation example:

\[
\frac{p \to p}{p^+ \to p^+} \quad \frac{p, p^+ \to p^+}{p^+ \to p} (\text{cut})
\]

\[
\frac{p \to p}{p, p^+ \to p} \quad \frac{p^+ \to p}{p^+ \to p}
\]

- $L_\infty^+$ is equivalent to $L_\omega^+$.
- Work in progress: cut elimination in $L_\infty^+$, cf. Savateev’s talk today.
System with Circular Proofs, $L_{circ}^+$

\[
\begin{align*}
\Pi \rightarrow A & \\ \Pi & \rightarrow A^+ \\
\Pi_1 \rightarrow A & \quad \Pi_2 \rightarrow A^+ \\ \Pi_1, \Pi_2 & \rightarrow A^+ \\
\Gamma, A, \Delta \rightarrow C & \quad \Gamma, A, A^+, \Delta \rightarrow C \\
\Gamma, A^+, \Delta & \rightarrow C
\end{align*}
\]

We allow to use the conclusion of the negative rule to be used as a premise in its derivation tree (backlink).
System with Circular Proofs, $\mathbb{L}_{\text{circ}}^+$

\[
\begin{align*}
\Pi &\rightarrow A & \Pi_1 &\rightarrow A & \Pi_2 &\rightarrow A^+ & \Gamma, A, \Delta &\rightarrow C \\
\Pi &\rightarrow A^+ & \Pi_1, \Pi_2 &\rightarrow A^+ & \Gamma, A^+, \Delta &\rightarrow C
\end{align*}
\]

We allow to use the conclusion of the negative rule to be used as a premise in its derivation tree (\textit{backlink}).

Example:

\[
\begin{align*}
p &\rightarrow p & p, (p \setminus p)^+ &\rightarrow p \\
p, p \setminus p &\rightarrow p & p, p \setminus p, (p \setminus p)^+ &\rightarrow p
\end{align*}
\]

\[
\begin{align*}
p, (p \setminus p)^+ &\rightarrow p \\
(p \setminus p)^+ &\rightarrow p \setminus p
\end{align*}
\]

Correctness condition: in each backlink, the active occurrence $A^+$ in the premise should be tracked down to the same active occurrence in the goal.

The circular system (with cut) is equivalent to Pratt's axiomatisation for general action algebras.

System with Circular Proofs, \( L_{circ}^+ \)

\[
\frac{\Pi \rightarrow A}{\Pi \rightarrow A^+} \quad \frac{\Pi_1 \rightarrow A \quad \Pi_2 \rightarrow A^+}{\Pi_1, \Pi_2 \rightarrow A^+} \quad \frac{\Gamma, A, \Delta \rightarrow C}{\Gamma, A^+, \Delta \rightarrow C} \quad \frac{\Gamma, A, A^+, \Delta \rightarrow C}{\Gamma, A^+, \Delta \rightarrow C}
\]

We allow to use the conclusion of the negative rule to be used as a premise in its derivation tree (backlink).

Example:

\[
\frac{p \rightarrow p, p \rightarrow p \rightarrow p}{p \rightarrow p, p \rightarrow p, (p \rightarrow p)^+ \rightarrow p \rightarrow p} \quad \frac{p \rightarrow p, p \rightarrow p \rightarrow p}{p \rightarrow p, p \rightarrow p, (p \rightarrow p)^+ \rightarrow p \rightarrow p}
\]

\[
\frac{p \rightarrow p \rightarrow p \rightarrow p}{p \rightarrow p, (p \rightarrow p)^+ \rightarrow p \rightarrow p} \quad \frac{p \rightarrow p \rightarrow p \rightarrow p}{(p \rightarrow p)^+ \rightarrow p \rightarrow p}
\]

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System with Circular Proofs, $\mathbf{L}^+_{\text{circ}}$

\[
\begin{align*}
\frac{\Pi \to A}{\Pi \to A^+} & \quad \frac{\Pi_1 \to A \quad \Pi_2 \to A^+}{\Pi_1, \Pi_2 \to A^+} \quad \frac{\Gamma, A, \Delta \to C \quad \Gamma, A, A^+, \Delta \to C}{\Gamma, A^+, \Delta \to C}
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System with Circular Proofs, $\mathbf{L}^+_{\text{circ}}$

\[
\frac{\Pi \rightarrow A}{\Pi \rightarrow A^+} \quad \frac{\Pi_1 \rightarrow A \quad \Pi_2 \rightarrow A^+}{\Pi_1, \Pi_2 \rightarrow A^+} \quad \frac{\Gamma, A, \Delta \rightarrow C \quad \Gamma, A, A^+, \Delta \rightarrow C}{\Gamma, A^+, \Delta \rightarrow C}
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System with Circular Proofs, $\text{L}_{\text{circ}}^+$

\[
\begin{array}{c}
\Pi \rightarrow A \\
\hline
\Pi \rightarrow A^+ \\
\end{array} \quad \begin{array}{c}
\Pi_1 \rightarrow A \\
\Pi_2 \rightarrow A^+ \\
\hline
\Pi_1, \Pi_2 \rightarrow A^+ \\
\end{array} \quad \begin{array}{c}
\Gamma, A, \Delta \rightarrow C \\
\hline
\Gamma, A^+, \Delta \rightarrow C \\
\end{array}
\]

We allow to use the conclusion of the negative rule to be used as a premise in its derivation tree (backlink).

Correctness condition: in each backlink, the active occurrence $A^+$ in the premise should be tracked down to the same active occurrence in the goal.

The circular system (with cut) is equivalent to Pratt’s axiomatisation for general action algebras.

Language Interpretation

\[ w(A) \subseteq \Sigma^+ \]

\[ w(A \setminus B) = w(A) \setminus w(B) = \{ u \in \Sigma^+ \mid (\forall v \in w(A)) \ vu \in w(B) \} \]

\[ w(B / A) = w(B) / w(A) = \{ u \in \Sigma^+ \mid (\forall v \in w(A)) \ uv \in w(B) \} \]

\[ w(A \cdot B) = w(A) \cdot w(B) = \{ uv \mid u \in w(A), v \in w(B) \} \]

\[ w(A^+) = \{ u_1 \ldots u_n \mid u_i \in w(A), n \geq 1 \} \]
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Theorem (M. Pentus 1995)

\[ L \vdash A \rightarrow B \iff w(A) \subseteq w(B) \text{ for all } w. \]
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**Open question:** completeness of \( L^+_\omega \).
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Theorem (M. Pentus 1995)

\( \mathbf{L} \vdash A \rightarrow B \iff w(A) \subseteq w(B) \) for all \( w \).

Open question: completeness of \( \mathbf{L}_{\omega}^+ \).

A partial result [Ryzhkova 2013]: completeness for the fragment without \( \cdot \), where \( ^+ \) appears only as \( A^+ \setminus B \) or \( B / A^+ \).
The exponential, $\!$, governed by the following rules:

\[
\begin{align*}
\Gamma, A, \Delta &\rightarrow C & !A_1, \ldots, !A_n &\rightarrow B & \Gamma, \Delta &\rightarrow C \\
\Gamma, !A, \Delta &\rightarrow C & !A_1, \ldots, !A_n &\rightarrow !B & \Gamma, !A, \Delta &\rightarrow C \\
\Gamma, \Phi, !A, \Delta &\rightarrow C & !A_1, \ldots, !A_n &\rightarrow !B & \Gamma, !A, \Phi, \Delta &\rightarrow C \\
\Gamma, !A, \Phi, \Delta &\rightarrow C & \Gamma, \Phi, !A, \Delta &\rightarrow C & \Gamma, !A, !A, \Delta &\rightarrow C \\
\Gamma, !A, \Phi, \Delta &\rightarrow C & \Gamma, \Phi, !A, \Delta &\rightarrow C & \Gamma, !A, \Delta &\rightarrow C
\end{align*}
\]

allows encoding *derivation from a finite theory* as a derivation of one formula:

\[A_1 \rightarrow B_1, \ldots, A_k \rightarrow B_k \vdash \Gamma \rightarrow C \iff ! (A_1 \setminus B_1), \ldots, ! (A_k \setminus B_k), \Gamma \rightarrow C\]
Even More Complexity: Iteration and the Exponential

The exponential, $!$, governed by the following rules:

\[
\begin{align*}
\frac{\Gamma, A, \Delta \rightarrow C}{\Gamma, !A, \Delta \rightarrow C} & \quad \frac{!A_1, \ldots, !A_n \rightarrow B}{!A_1, \ldots, !A_n \rightarrow !B} & \frac{\Gamma, \Delta \rightarrow C}{\Gamma, !A, \Delta \rightarrow C} \\
\frac{\Gamma, \Phi, !A, \Delta \rightarrow C}{\Gamma, !A, \Phi, \Delta \rightarrow C} & \frac{\Gamma, !A, \Phi, \Delta \rightarrow C}{\Gamma, \Phi, !A, \Delta \rightarrow C} & \frac{\Gamma, !A, !A, \Delta \rightarrow C}{\Gamma, !A, \Delta \rightarrow C}
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Therefore,

- \textbf{L} with $!$ is undecidable ($\Sigma^0_1$-complete): encoding derivations in semi-Thue systems (actually a subset of rules for $!$ is sufficient, see Scedrov’s talk today);
Even More Complexity: Iteration and the Exponential

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\Gamma, \text{!}A, \Delta & \rightarrow C & \text{!}A_1, \ldots, \text{!}A_n & \rightarrow \text{!}B & \Gamma, \text{!}A, \Delta & \rightarrow C \\
\Gamma, \Phi, \text{!}A, \Delta & \rightarrow C & \Gamma, \text{!}A, \Phi, \Delta & \rightarrow C & \Gamma, \text{!}A, \text{!}A, \Delta & \rightarrow C \\
\Gamma, \text{!}A, \Phi, \Delta & \rightarrow C & \Gamma, \Phi, \text{!}A, \Delta & \rightarrow C & \Gamma, \Phi, \text{!}A, \Delta & \rightarrow C \\
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Therefore,

- \text{L with !} is undecidable (\(\Sigma^0_1\)-complete): encoding derivations in semi-Thue systems (actually a subset of rules for ! is sufficient, see Scedrov’s talk today);

- \text{L with ! and * is } \Pi^1_1\text{-hard: encoding Kozen 2002 (deriving Horn clauses in *-continuous Kleene algebra is } \Pi^1_1\text{-complete}).

Open question: \(\Pi^1_1\) upper bound.
Thanks+