Prolongable Satisfaction Classes and Iterations of Uniform Reflection over PA

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  - a proof of conservativity of $\mathrm{CS}_0 + \mathrm{B}\Sigma_1(S)$ over $\mathcal{UR}^{\omega}(\mathrm{PA})$.
- The methods essentially consists in arithmetizing some well-known proofs.

Full satisfaction class

Let $\mathcal{M} \models \mathsf{PA}$ and S be a fresh binary predicate. A *full satisfation class* on $\mathcal{M}$ is a set $\mathbf{S} \subseteq M^2$ such that the following $\mathcal{L}_{\mathsf{PA}} \cup \{S\}$ sentences are true in $(\mathcal{M}, \mathbf{S})$:

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The theory consisting of PA and the above axioms is called CS^-.

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Partial inductive satisfaction class

Let $\mathcal{M} \models \mathrm{PA}$. Let S be a fresh binary predicate and c a nonstandard element of $\mathcal{M}$. A *partial, inductive satisfaction class* (c-restricted) on $\mathcal{M}$ is a set $\mathbf{S}_c \subseteq M^2$ such that the following $\mathcal{L}_{\mathrm{PA}} \cup \{S\}$ sentences are true in $(\mathcal{M}, \mathbf{S}_c)$ ($\phi \in \mathrm{compl}(c)$ means that ϕ is of complexity (logical depth) at most c):

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5. for every $\phi(x, \bar{y})$ in $\mathcal{L}_{\mathrm{PA}} \cup \{S\}$, the instantiation of induction scheme with $\phi(x, \bar{y})$

Iterations of Uniform Reflection

Definition

Let Th be a Δ_1 definable arithmetical theory. Define

$$\mathcal{UR}(\mathsf{Th}) \quad = \quad \mathsf{Th} + \{\forall \bar{x} \;\; (\mathsf{Pr}_{\mathsf{Th}}(\ulcorner \phi(\underline{\bar{x}}) \urcorner) \rightarrow \phi(\bar{x})) \;\mid\; \phi(\bar{x}) \in \mathcal{L}_{\mathsf{PA}}\}$$

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\mathcal{UR}^\omega(\mathrm{PA}) &= \bigcup_{n \in \omega} \mathcal{UR}^n(\mathrm{PA})
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Some history and context

Theorem (Enayat, Visser)

If $(\mathcal{M}, \mathbf{S}_c)$ *is a model of* PA *with a partial inductive satisfaction class, then there exist* $\mathcal{M} \precneqq_e \mathcal{N}$ *and a full satisfaction class* $\mathbf{S}'$ *on* $\mathcal{N}$ *such that* $\mathbf{S} \subseteq \mathbf{S}'$*.*

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However: if $\mathbf{S}$ is a (full or partial inductive) satisfaction class on $\mathcal{M}$, then $\mathcal{M}$ is recursively saturated (Lachlan's Theorem). If $\mathcal{M}$ is rather classless, then it does not admit a recursively saturated end-extension.

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The proof uses countable, recursively saturated models and mimics Henkin's proof of Completeness Theorem. It shows conservativity over ω iterations of (internal) ω-rule. That $\mathrm{CS}_0 \vdash \mathcal{UR}^{\omega}(\mathrm{PA})$ was first shown by Bartosz Wcisło.

Some history and context, continued

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for $\phi \in \mathcal{L}_{\mathsf{PA}}$ and full induction for S. Then use overspill over $\mathbb{N}$ on a formula "S is compositional for all formulae of complexity at most n." □

Some context and history, continued, continued

Ending this section let me give two metamathematical properties of CS_0.

- CS_0 can be alternatively axiomatized by taking CS^- with the global reflection principle, i.e.

$$\forall\phi\forall\alpha \in \mathsf{Asn}(\phi) \quad (\mathsf{Pr}_{\mathsf{PA}}(\phi) \rightarrow S(\phi,\alpha)).$$

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- CS_0 has super-exponential speed-up over $\mathcal{UR}^\omega(\mathsf{PA})$. Consequently, it is non-interpretable in it.

Limitations

If $(\mathcal{N}, \mathbf{S}) \models \mathsf{CS}_0$, then also

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Even in models of decent theories by far not every satisfaction class has this property.

Key definition

Let $\mathcal{M} \models \mathrm{PA}$, $c \in \mathcal{M} \setminus \mathbb{N}$ and $\mathbf{S}_c$ be a partial inductive satisfaction class.
We say that $\mathbf{S}_c$ is prolongable (1-prolongable) if there exist

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$$\{S'(\phi, \alpha) \ \mid \ S(\phi, \alpha)\} + \{\text{"}S' \text{ is compositional on } \phi\text{"} \ \mid \ \phi \in \mathcal{L}_{\mathrm{PA}}\}.$$

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and the same holds in $(\mathcal{M}, \mathbf{S}_{c'})$. □

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$(2) \Rightarrow (3)$.

Having $\mathcal{M} = \mathcal{M}_0 \precneqq_e \mathcal{M}_1 \precneqq M_2 \ldots$ *and* $\mathbf{S}_d = \mathbf{S}_{d_0} \subseteq \mathbf{S}_{d_1} \subseteq \mathbf{S}_{d_2}$ *define*

$$\mathcal{N} = \bigcup_i \mathcal{M}_i \qquad\qquad \mathbf{S} = \bigcup_i \mathbf{S}_{d_i}$$

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$$\underset{d_0 \in M_0}{\mathcal{M}_0} \precneqq_e \cdots \precneqq_e \underset{d_n \in M_n \setminus M_{n-1}}{\mathcal{M}_n} \precneqq_e \cdots \underset{d_\omega \in M_\omega \setminus \bigcup_{i \in \omega} M_i}{\mathcal{M}_\omega} \precneqq_e \mathcal{M}_{\omega+1} \cdots,$$

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$$(\mathcal{M}_\omega, \mathbf{S}_\omega) \models \mathrm{B}\Sigma_1(S).$$

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Longer chains of satisfaction classes?

Thanks to Barwise-Schlipf-Ressayre theorem (countable recursively saturated models are resplendent) over an arbitrary countable recursively saturated model of $\mathcal{UR}^{\omega}(\mathrm{PA})$ and for arbitrary $\alpha < \omega_1$ we can find a chain of satisfaction classes of length α

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The existence of ω_1 such chain is equivalent to $\mathcal{M}$ satisfying the arithmetical consequences of CS.

Thank you for your attention.