FDE-Modalities and Weak Definability

Sergei P. Odintsov[1]

[1] Sobolev Institute of Mathematics (Novosibirsk, Russia)
odintsov@math.nsc.ru

(joint work with Heinrich Wansing)

Wormshop, 15–20.10.2017

Logic R of relevant implication

1	$\varphi \to \varphi$	Self-implication
2	$(\varphi \to \psi) \to ((\chi \to \varphi) \to (\chi \to \psi))$	Prefixing
3	$(\varphi \to (\varphi \to \psi)) \to (\varphi \to \psi)$	Contraction
4	$(\varphi \to (\psi \to \chi)) \to (\psi \to (\varphi \to \chi))$	Permutation
5	$(\varphi \wedge \psi) \to \varphi$, $(\varphi \wedge \psi) \to \psi$	$\wedge$-Elimination
6	$((\varphi \to \psi) \wedge (\varphi \to \chi)) \to (\varphi \to (\varphi \wedge \psi)$	$\wedge$-Introduction
7	$\varphi \to (\varphi \vee \psi)$, $\psi \to (\varphi \vee \psi)$	$\vee$-Introduction
8	$((\varphi \to \chi) \wedge (\psi \to \chi)) \to ((\varphi \vee \psi) \to \chi)$	$\vee$-Elimination
9	$(\varphi \wedge (\psi \vee \chi)) \to ((\varphi \wedge \psi) \vee \chi)$	Distribution
10	$(\varphi \to \sim\varphi) \to \sim\varphi$	Reductio
11	$(\varphi \to \sim\psi) \to (\psi \to \sim\varphi)$	Contraposition
12	$\sim\sim\varphi \to \varphi$	Double negation

Rules: *Modus ponens* and *Adjunction*

FDE is a first degree fragment of R

- (Variable Sharing Principle)

  If $\varphi \to \psi$ is a theorem of R,
  then φ and ψ have a common variable.

- For $\to$-free φ and ψ,
  $\varphi \vdash_{\mathsf{FDE}} \psi$ iff $\varphi \to \psi$ is a theorem of R

$$\mathbf{B4} := \langle \{\mathbf{T}, \mathbf{F}, \mathbf{N}, \mathbf{B}\}, \wedge, \vee, \sim, \{\mathbf{T}, \mathbf{B}\} \rangle$$

$$\mathbf{B3} := \langle \{\mathbf{T}, \mathbf{F}, \mathbf{N}\}, \wedge, \vee, \neg, \{\mathbf{T}\} \rangle$$

Elements of **B4** are subsets of $\{0, 1\}$:

$$\mathbf{T} = \{1\},\ \mathbf{F} = \{0\},\ \mathbf{N} = \emptyset,\ \mathbf{B} = \{0, 1\},$$

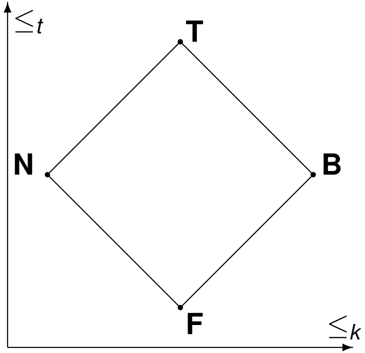
then matrix operations are operations on sets classical truth values, eg.

$$\{0, 1\} \vee \{0\} = \{0, 1\}, \{0, 1\} \vee \emptyset = \{1\}, \sim \{0, 1\} = \{0, 1\}.$$

As a result we obtain lattice operations wrt truth ordering.

B4 as a bilattice.

$\leq_t$ is the truth (logical) ordering and $\leq_k$ is the knowledge (information) ordering



B4 and First Degree Entailment

- $\varphi \models_{\mathbf{B4}} \psi$ iff $\forall v : Prop \to \{\mathbf{T}, \mathbf{F}, \mathbf{N}, \mathbf{B}\} \quad v(\varphi) \leq_t v(\psi)$
- [Dunn 76] $\varphi \vdash_{\mathrm{FDE}} \psi$ iff $\varphi \models_{\mathbf{B4}} \psi$

B4 and First Degree Entailment

- $\varphi \models_{\mathbf{B4}} \psi$ iff $\forall v : Prop \rightarrow \{\mathbf{T}, \mathbf{F}, \mathbf{N}, \mathbf{B}\} \quad v(\varphi) \leq_t v(\psi)$
- [Dunn 76] $\varphi \vdash_{\mathrm{FDE}} \psi$ iff $\varphi \models_{\mathbf{B4}} \psi$

FDE sequent calculus

1. Sequents: $\varphi \vdash \psi$
2. Axioms:
   - $\varphi \vdash \varphi$
   - $\varphi \wedge \psi \vdash \varphi \quad \varphi \wedge \psi \vdash \psi$
   - $\varphi \vdash \varphi \vee \psi \quad \psi \vdash \varphi \vee \psi$
   - $\varphi \wedge (\psi \vee \chi) \vdash (\varphi \wedge \psi) \vee \chi$
   - $\varphi \vdash {\sim}{\sim}\varphi \quad {\sim}{\sim}\varphi \vdash \varphi$
3. Rules:

$$\frac{\varphi \vdash \psi \quad \varphi \vdash \chi}{\varphi \vdash \psi \wedge \chi} \qquad \frac{\varphi \vdash \chi \quad \psi \vdash \chi}{\varphi \vee \psi \vdash \chi} \qquad \frac{\varphi \vdash \psi \quad \psi \vdash \chi}{\varphi \vdash \chi}$$

$$\frac{\varphi \vdash \psi}{{\sim}\psi \vdash {\sim}\varphi}$$

Adding weak implication: B4 as a twist-structure.

- Represent elements S of **B4** as characteristic functions of subsets of $\{0,1\}$, i.e., as pairs $S = (a, b)$, where
  $$a = 1 \text{ iff } 1 \in S \text{ and } b = 1 \text{ iff } 0 \in S.$$
  $$\mathbf{T} = (1,0),\ \mathbf{F} = (0,1),\ \mathbf{N} = (0,0),\ \mathbf{B} = (1,1).$$
- Matrix operations of **B4** as twist-operations:
  $$(a,b) \vee (c,d) = (a \vee c, b \wedge d),\ \ (a,b) \wedge (c,d) = (a \wedge c, b \vee d),$$
  $$\sim(a,b) = (b,a).$$
- Implication operation on **B4**:
  $$(a,b) \to (c,d) = (a \to c, a \wedge d),$$
- Add the constant $\bot$ interpreted as **F** and consider Belnap's matrix in this extended language:
  $$\mathbf{B4}^{\to}_{\bot} := \langle\{\mathbf{T},\mathbf{F},\mathbf{N},\mathbf{B}\}, \wedge, \vee, \to, \bot, \sim, \{\mathbf{T},\mathbf{B}\}\rangle$$

Axiomatics of $\mathbf{B4}^{\rightarrow}$ and $\mathbf{B4}^{\rightarrow}_{\perp}$

- $LB4^{\rightarrow} = \{\varphi \mid \forall v(v(\varphi) \in \{\mathbf{T}, \mathbf{B}\})\}$
- Hilbert style calculus for $LB4^{\rightarrow}$
  - Axioms for positive fragment of classical logic
  - Strong negation axioms:
    N1. $\sim(\alpha \to \beta) \leftrightarrow \alpha \wedge \sim\beta$ **N2.** $\sim(\alpha \wedge \beta) \leftrightarrow \sim\alpha \vee \sim\beta$
    N3. $\sim\sim\alpha \leftrightarrow \alpha$ **N4.** $\sim(\alpha \vee \beta) \leftrightarrow \sim\alpha \wedge \sim\beta$.
  - Inference rule:

$$MP \quad \frac{\alpha, \ \alpha \to \beta}{\beta}$$

- $LB4^{\rightarrow}_{\perp} = LB4^{\rightarrow} + \{\perp \to p, \ p \to \sim\perp\}$

Axiomatics of $\mathbf{B4}^{\rightarrow}$ and $\mathbf{B4}^{\rightarrow}_{\perp}$

- $L\mathbf{B4}^{\rightarrow} = \{\varphi \mid \forall v(v(\varphi) \in \{\mathbf{T}, \mathbf{B}\})\}$
- Hilbert style calculus for $L\mathbf{B4}^{\rightarrow}$
  - Axioms for positive fragment of classical logic
  - Strong negation axioms:
    N1. $\sim(\alpha \rightarrow \beta) \leftrightarrow \alpha \wedge \sim\beta$ **N2.** $\sim(\alpha \wedge \beta) \leftrightarrow \sim\alpha \vee \sim\beta$
    N3. $\sim\sim\alpha \leftrightarrow \alpha$ **N4.** $\sim(\alpha \vee \beta) \leftrightarrow \sim\alpha \wedge \sim\beta$.
  - Inference rule:

$$MP \quad \frac{\alpha, \ \alpha \rightarrow \beta}{\beta}$$

- $L\mathbf{B4}^{\rightarrow}_{\perp} = L\mathbf{B4}^{\rightarrow} + \{\perp \rightarrow p, \ p \rightarrow \sim\perp\}$

Axiomatics of $\mathbf{B4}^{\rightarrow}$ and $\mathbf{B4}_{\perp}^{\rightarrow}$

- $L\mathbf{B4}^{\rightarrow} = \{\varphi \mid \forall v(v(\varphi) \in \{\mathbf{T}, \mathbf{B}\})\}$
- Hilbert style calculus for $L\mathbf{B4}^{\rightarrow}$
  - Axioms for positive fragment of classical logic
  - Strong negation axioms:
    N1. $\sim(\alpha \rightarrow \beta) \leftrightarrow \alpha \wedge \sim\beta$ **N2.** $\sim(\alpha \wedge \beta) \leftrightarrow \sim\alpha \vee \sim\beta$
    N3. $\sim\sim\alpha \leftrightarrow \alpha$ **N4.** $\sim(\alpha \vee \beta) \leftrightarrow \sim\alpha \wedge \sim\beta$.
  - Inference rule:
    $$MP \quad \frac{\alpha, \ \alpha \rightarrow \beta}{\beta}$$
- $L\mathbf{B4}_{\perp}^{\rightarrow} = L\mathbf{B4}^{\rightarrow} + \{\perp \rightarrow p, \ p \rightarrow \sim\perp\}$

Adding strong implication: Brady's BN4

- [Ross Brady 82] BN4 $= L\mathbf{B4}^{\Rightarrow}$,

  $$\text{where } x \Rightarrow y := (x \to y) \vee (\sim y \to \sim x).$$

  "the most natural truth-functional conditional associated with FDE" [B. Meier, J. Slaney]

- Weak implication via strong implication [Arieli & Avron 96]

  $$x \to y := (x \Rightarrow (x \Rightarrow y)) \vee y$$

- Strong implication is substructural

  $$x \Rightarrow (x \Rightarrow y) \neq x \Rightarrow y$$

- $\mathbf{B3}^{\Rightarrow} = Ł_3$

Adding strong implication: Brady's BN4

- [Ross Brady 82] BN4 $= L\mathbf{B4}^{\Rightarrow}$,

  where $x \Rightarrow y := (x \to y) \vee (\sim y \to \sim x)$.

  "the most natural truth-functional conditional associated with FDE" [B. Meier, J. Slaney]

- Weak implication via strong implication [Arieli & Avron 96]

$$x \to y := (x \Rightarrow (x \Rightarrow y)) \vee y$$

- Strong implication is substructural

$$x \Rightarrow (x \Rightarrow y) \neq x \Rightarrow y$$

- $\mathbf{B3}^{\Rightarrow} = \mathbf{Ł}_3$

Adding strong implication: Brady's BN4

- [Ross Brady 82] BN4 $= L\mathbf{B4}^{\Rightarrow}$,

  where $x \Rightarrow y := (x \to y) \vee (\sim y \to \sim x)$.

  "the most natural truth-functional conditional associated with FDE" [B. Meier, J. Slaney]

- Weak implication via strong implication [Arieli & Avron 96]

$$x \to y := (x \Rightarrow (x \Rightarrow y)) \vee y$$

- Strong implication is substructural

$$x \Rightarrow (x \Rightarrow y) \neq x \Rightarrow y$$

- $\mathbf{B3}^{\Rightarrow} = \text{Ł}_3$

Adding strong implication: Brady's BN4

- [Ross Brady 82] BN4 $= L\mathbf{B4}^{\Rightarrow}$,

  where $x \Rightarrow y := (x \rightarrow y) \vee (\sim y \rightarrow \sim x)$.

  "the most natural truth-functional conditional associated with FDE" [B. Meier, J. Slaney]

- Weak implication via strong implication [Arieli & Avron 96]

$$x \rightarrow y := (x \Rightarrow (x \Rightarrow y)) \vee y$$

- Strong implication is substructural

$$x \Rightarrow (x \Rightarrow y) \neq x \Rightarrow y$$

- $\mathbf{B3}^{\Rightarrow} = \L_3$

Adding strong implication: Brady's BN4

- [Ross Brady 82] BN4 $= L\mathbf{B4}^{\Rightarrow}$,

  where $x \Rightarrow y := (x \to y) \vee ({\sim}y \to {\sim}x)$.

  "the most natural truth-functional conditional associated with FDE" [B. Meier, J. Slaney]

- Weak implication via strong implication [Arieli & Avron 96]

$$x \to y := (x \Rightarrow (x \Rightarrow y)) \vee y$$

- Strong implication is substructural

$$x \Rightarrow (x \Rightarrow y) \neq x \Rightarrow y$$

- $\mathbf{B3}^{\Rightarrow} = \text{Ł}_3$

Axiomatics of BN4

- Axioms

  $p \Rightarrow p$

  $(p \wedge q) \Rightarrow p, \ (p \wedge q) \Rightarrow q$

  $((p \Rightarrow q) \wedge (p \Rightarrow r)) \Rightarrow (p \Rightarrow (q \wedge r))$

  $(p \wedge (q \vee r)) \Rightarrow ((p \vee q) \wedge (p \vee r))$

  $(p \Rightarrow q) \Rightarrow ({\sim}q \Rightarrow {\sim}p)$

  ${\sim}{\sim}p \Rightarrow p$

  ${\sim}p \Rightarrow (p \vee (p \Rightarrow q))$

  $p \vee {\sim}q \vee (p \Rightarrow q)$

  $(p \Rightarrow p) \Rightarrow ({\sim}p \Rightarrow {\sim}p))$

  $p \vee (({\sim}p \Rightarrow p) \Rightarrow q)$

  $(p \vee q) \Leftrightarrow {\sim}({\sim}p \wedge {\sim}q)$

- Rules

$$\frac{p, q}{p \wedge q}, \quad \frac{p, p \Rightarrow q}{q}, \quad \frac{p \Rightarrow q, r \Rightarrow t}{(q \Rightarrow r) \Rightarrow (p \Rightarrow t)}, \quad \frac{r \vee p, r \vee (p \Rightarrow q)}{r \vee q}$$

From $\mathbf{B3}^{\rightarrow}$, $\mathbf{B4}^{\rightarrow}$, $\mathbf{B4}^{\rightarrow}_{\perp}$ to Nelson's **N3**, **N4**, $\mathbf{N4}^{\perp}$

- $L\mathbf{B3}^{\rightarrow} = L\mathbf{B4}^{\rightarrow} + \{\sim p \rightarrow (p \rightarrow q)\}$
- Axiomatics: replace

  "Axioms for positive fragment of classical logic"

  by

  "Axioms for positive fragment of intuitionistic logic"

- $L\mathbf{B3}^{\rightarrow} = \mathbf{N3} + \{p \vee (p \rightarrow q)\}$,

  $L\mathbf{B4}^{\rightarrow} = \mathbf{N4} + \{p \vee (p \rightarrow q)\}$,

  $L\mathbf{B4}^{\rightarrow}_{\perp} = \mathbf{N4}^{\perp} + \{p \vee (p \rightarrow q)\}$

Possible World Semantics for **N3**,**N4**, and **N4**$^{\perp}$

- **N4**-model is $\langle W, \leq, V\rangle$, where $V : Prop \times W \to$ **B4** and
$$w \leq w' \;\Rightarrow\; V(p, w) \leq_k V(p, w')$$
- **N3**-model is $\langle W, \leq, V\rangle$, where $V : Prop \times W \to$ **B3** and
$$w \leq w' \;\Rightarrow\; V(p, w) \leq_k V(p, w')$$

Possible World Semantics for **N3**, **N4**, and $\mathbf{N4}^{\perp}$

$$
\begin{array}{lcl}
V(\varphi \vee \psi, w) & = & V(\varphi, w) \vee V(\psi, w) \\
V(\varphi \wedge \psi, w) & = & V(\varphi, w) \wedge V(\psi, w) \\
V(\sim \varphi, w) & = & \sim V(\varphi, w) \\
1 \in V(\varphi \to \psi, w) & \text{iff} & \forall w' \geq w \ (1 \in V(\varphi, w') \Rightarrow 1 \in V(\psi, w')) \\
0 \in V(\varphi \to \psi, w) & \text{iff} & 1 \in V(\varphi, w) \text{ and } 0 \in V(\psi, w)
\end{array}
$$

- $V(\perp, w) = \mathbf{F}$ in case of $\mathbf{N4}^{\perp}$

Possible World Semantics for **N3**, **N4** and **N4**$^{\perp}$

- $\mathcal{M} \models \varphi$ iff $1 \in V(\varphi, w)$ for all $w \in W$
- $\mathcal{M}, w \models \Gamma$ iff $\mathcal{M}, w \models \varphi$ for all $\varphi \in \Gamma$
- $\Gamma \models_{\mathbf{N4}} \varphi$ iff $\forall$ **N4**-model $\mathcal{M} \forall w \, (\mathcal{M}, w \models \Gamma \Rightarrow \mathcal{M}, w \models \varphi)$

  $\Gamma \models_{\mathbf{N3}} \varphi$ iff $\forall$ **N3**-model $\mathcal{M} \forall w \, (\mathcal{M}, w \models \Gamma \Rightarrow \mathcal{M}, w \models \varphi)$
- **N3**, **N4** and **N4**$^{\perp}$ are strongly complete w.r.t. respective classes of models

Replacement in Nelson logics

- Replacement rule fails for **N3**, **N4**, and **N4**$^\perp$

$$\frac{\varphi \leftrightarrow \psi}{\chi(\varphi) \leftrightarrow \chi(\psi)}$$

- $\sim (\varphi \to \psi) \leftrightarrow (\varphi \wedge \sim \psi) \in$ **N4**. Let $\chi(p) = \sim p$.

$$(\varphi \to \psi) \leftrightarrow (\sim \varphi \vee \psi) \notin \textbf{N4}$$

- Positive replacement rule holds for Nelson logics

$$\frac{\varphi \leftrightarrow \psi}{\chi(\varphi) \leftrightarrow \chi(\psi)},$$

  where $\chi(p)$ is $\sim$-free
- Weak replacement rule holds for Nelson logics

$$\frac{\varphi \leftrightarrow \psi \quad \sim \varphi \leftrightarrow \sim \psi}{\chi(\varphi) \leftrightarrow \chi(\psi)}$$

Replacement in Nelson logics

- Replacement rule fails for **N3**, **N4**, and **N4**$^\perp$

$$\frac{\varphi \leftrightarrow \psi}{\chi(\varphi) \leftrightarrow \chi(\psi)}$$

- $\sim (\varphi \to \psi) \leftrightarrow (\varphi \wedge \sim \psi) \in$ **N4**. Let $\chi(p) = \sim p$.

$$(\varphi \to \psi) \leftrightarrow (\sim \varphi \vee \psi) \notin \textbf{N4}$$

- Positive replacement rule holds for Nelson logics

$$\frac{\varphi \leftrightarrow \psi}{\chi(\varphi) \leftrightarrow \chi(\psi)},$$

where $\chi(p)$ is $\sim$-free

- Weak replacement rule holds for Nelson logics

$$\frac{\varphi \leftrightarrow \psi \quad \sim \varphi \leftrightarrow \sim \psi}{\chi(\varphi) \leftrightarrow \chi(\psi)}$$

Replacement in Nelson logics

- Replacement rule fails for **N3**, **N4**, and $\mathbf{N4}^{\perp}$

$$\frac{\varphi \leftrightarrow \psi}{\chi(\varphi) \leftrightarrow \chi(\psi)}$$

- $\sim(\varphi \to \psi) \leftrightarrow (\varphi \wedge \sim\psi) \in$ **N4**. Let $\chi(p) = \sim p$.

$$(\varphi \to \psi) \leftrightarrow (\sim\varphi \vee \psi) \notin \mathbf{N4}$$

- Positive replacement rule holds for Nelson logics

$$\frac{\varphi \leftrightarrow \psi}{\chi(\varphi) \leftrightarrow \chi(\psi)},$$

where $\chi(p)$ is $\sim$-free

- Weak replacement rule holds for Nelson logics

$$\frac{\varphi \leftrightarrow \psi \quad \sim\varphi \leftrightarrow \sim\psi}{\chi(\varphi) \leftrightarrow \chi(\psi)}$$

Replacement in Nelson logics

- Replacement rule fails for **N3**, **N4**, and $\mathbf{N4}^{\perp}$

$$\frac{\varphi \leftrightarrow \psi}{\chi(\varphi) \leftrightarrow \chi(\psi)}$$

- $\sim(\varphi \to \psi) \leftrightarrow (\varphi \wedge \sim \psi) \in$ **N4**. Let $\chi(p) = \sim p$.

$$(\varphi \to \psi) \leftrightarrow (\sim \varphi \vee \psi) \notin \mathbf{N4}$$

- Positive replacement rule holds for Nelson logics

$$\frac{\varphi \leftrightarrow \psi}{\chi(\varphi) \leftrightarrow \chi(\psi)},$$

  where $\chi(p)$ is $\sim$-free
- Weak replacement rule holds for Nelson logics

$$\frac{\varphi \leftrightarrow \psi \quad \sim \varphi \leftrightarrow \sim \psi}{\chi(\varphi) \leftrightarrow \chi(\psi)}$$

The basic FDE-modal logic **BK**

- The language $\mathcal{L}^m = \{\vee, \wedge, \to, \bot, \sim, \Box, \Diamond\}$.
- A **BK**-model is a tuple $\mathcal{M} = \langle W, R, V\rangle$, where W is a set of possible worlds, $R \subseteq W^2$ is an accessibility relation on W, and $V : Prop \times W \to \mathbf{B4}^{\to}_{\bot}$.
- V extends to non-modal formulas as follows:
  - $V(\varphi \vee \psi, w) = V(\varphi, w) \vee V(\psi, w)$;
  - $V(\varphi \wedge \psi, w) = V(\varphi, w) \wedge V(\psi, w)$;
  - $V(\varphi \to \psi, w) = V(\varphi, w) \to V(\psi, w)$;
  - $V({\sim}\varphi, w) = {\sim}V(\varphi, w)$;
  - $V(\bot, w) = \mathbf{F}$.

The basic FDE-modal logic **BK**

- V extends to modal formulas according to [Fitting 91]:
  - $V(\Box\varphi, w) = \inf_{\leq_t}\{V(\varphi, u) \mid wRu\}$
  - $V(\Diamond\varphi, w) = \sup_{\leq_t}\{V(\varphi, u) \mid wRu\}$

Alternative presentation of **BK**-models

- $\mathcal{M} = \langle W, R, v^+, v^- \rangle$, where $v^+, v^- : Prop \to 2^W$ are two valuations. Given a **BK**-model $\mathcal{M}$, we define verification and falsification relations, $\models^+$ and $\models^-$, between worlds of $\mathcal{M}$ and formulas of the language $\mathcal{L}^m$:

- $\mathcal{M}, w \models^+ p \;\Leftrightarrow\; w \in v^+(p)$; $\mathcal{M}, w \models^- p \;\Leftrightarrow\; w \in v^-(p)$

- $\mathcal{M}, w \models^+ \varphi \wedge \psi \;\Leftrightarrow\; (\mathcal{M}, w \models^+ \varphi$ and $\mathcal{M}, w \models^+ \psi)$
  $\mathcal{M}, w \models^- \varphi \wedge \psi \;\Leftrightarrow\; (\mathcal{M}, w \models^- \varphi$ or $\mathcal{M}, w \models^- \psi)$

- $\mathcal{M}, w \models^+ \varphi \vee \psi \;\Leftrightarrow\; (\mathcal{M}, w \models^+ \varphi$ or $\mathcal{M}, w \models^+ \psi)$
  $\mathcal{M}, w \models^- \varphi \vee \psi \;\Leftrightarrow\; (\mathcal{M}, w \models^- \varphi$ and $\mathcal{M}, w \models^- \psi)$

- $\mathcal{M}, w \models^+ \varphi \to \psi \;\Leftrightarrow\; (\mathcal{M}, w \models^+ \varphi \Rightarrow \mathcal{M}, w \models^+ \psi)$
  $\mathcal{M}, w \models^- \varphi \to \psi \;\Leftrightarrow\; (\mathcal{M}, w \models^+ \varphi$ and $\mathcal{M}, w \models^- \psi)$

Alternative presentation of **BK**-models

- $\mathcal{M} = \langle W, R, v^+, v^- \rangle$, where $v^+, v^- : Prop \to 2^W$ are two valuations. Given a **BK**-model $\mathcal{M}$, we define verification and falsification relations, $\models^+$ and $\models^-$, between worlds of $\mathcal{M}$ and formulas of the language $\mathcal{L}^m$:

- $\mathcal{M}, w \models^+ p \;\Leftrightarrow\; w \in v^+(p)$; $\;\mathcal{M}, w \models^- p \;\Leftrightarrow\; w \in v^-(p)$

- $\mathcal{M}, w \models^+ \varphi \wedge \psi \;\Leftrightarrow\; (\mathcal{M}, w \models^+ \varphi$ and $\mathcal{M}, w \models^+ \psi)$
  $\mathcal{M}, w \models^- \varphi \wedge \psi \;\Leftrightarrow\; (\mathcal{M}, w \models^- \varphi$ or $\mathcal{M}, w \models^- \psi)$

- $\mathcal{M}, w \models^+ \varphi \vee \psi \;\Leftrightarrow\; (\mathcal{M}, w \models^+ \varphi$ or $\mathcal{M}, w \models^+ \psi)$
  $\mathcal{M}, w \models^- \varphi \vee \psi \;\Leftrightarrow\; (\mathcal{M}, w \models^- \varphi$ and $\mathcal{M}, w \models^- \psi)$

- $\mathcal{M}, w \models^+ \varphi \to \psi \;\Leftrightarrow\; (\mathcal{M}, w \models^+ \varphi \Rightarrow \mathcal{M}, w \models^+ \psi)$
  $\mathcal{M}, w \models^- \varphi \to \psi \;\Leftrightarrow\; (\mathcal{M}, w \models^+ \varphi$ and $\mathcal{M}, w \models^- \psi)$

Alternative presentation of **BK**-models

- $\mathcal{M}, w \not\models^{+} \bot, \quad \mathcal{M}, w \models^{-} \bot$
- $\mathcal{M}, w \models^{+} {\sim}\varphi \quad \Leftrightarrow \quad \mathcal{M}, w \models^{-} \varphi$
  $\mathcal{M}, w \models^{-} {\sim}\varphi \quad \Leftrightarrow \quad \mathcal{M}, w \models^{+} \varphi$
- $\mathcal{M}, w \models^{+} \Box\varphi \quad \Leftrightarrow \quad \forall u(wRu \;\Rightarrow \mathcal{M}, u \models^{+} \varphi)$
  $\mathcal{M}, w \models^{-} \Box\varphi \quad \Leftrightarrow \quad \exists u(wRu \text{ and } \mathcal{M}, u \models^{-} \varphi)$
- $\mathcal{M}, w \models^{+} \Diamond\varphi \quad \Leftrightarrow \quad \exists u(wRu \text{ and } \mathcal{M}, u \models^{+} \varphi)$
  $\mathcal{M}, w \models^{-} \Diamond\varphi \quad \Leftrightarrow \quad \forall u(wRu \;\Rightarrow \mathcal{M}, u \models^{-} \varphi)$

BK-valid formulas

- $\mathcal{M} = \langle W, R, V \rangle$ is a **BK**-model; φ is a formula.

$$\mathcal{M} \vDash \varphi \text{ iff } V(\varphi, w) \in \{\mathbf{T}, \mathbf{B}\} \text{ for all } w \in W \text{ iff}$$
$$\text{iff } \mathcal{M} \models^{+} \varphi \text{ for all } w \in W$$

- φ is **BK**-valid iff $\mathcal{M} \vDash \varphi$ for every **BK**-model $\mathcal{M}$
- All tautologies of **K** are **BK**-valid.
- The set of **BK**-valid formulas is not closed under the replacement rule:

  $\sim(p \to q) \leftrightarrow (p \wedge \sim q) \in$ **BK**, but

  $(p \to q) \leftrightarrow (\sim p \vee q) \notin$ **BK**.

Logic **BK**

BK is the least set of formulas closed under the rules of substitution, *modus ponens* and the monotonicity rules for both modalities; and containing the following axioms:

1. axioms of classical propositional logic in the language $\{\vee, \wedge, \to, \bot\}$;
2. strong negation axioms:

$$\begin{array}{ll} {\sim}{\sim}p \leftrightarrow p; & {\sim}(p \vee q) \leftrightarrow ({\sim}p \wedge {\sim}q); \\ {\sim}(p \wedge q) \leftrightarrow ({\sim}p \vee {\sim}q); & {\sim}(p \to q) \leftrightarrow (p \wedge {\sim}q); \end{array}$$

3. modal axioms:

$$\begin{array}{ll} (\Box p \wedge \Box q) \to \Box(p \wedge q); & \Box(p \to p); \\ \neg\Box p \leftrightarrow \Diamond\neg p; & \neg\Diamond p \leftrightarrow \Box\neg p; \\ \Box p \Leftrightarrow {\sim}\Diamond{\sim}p; & \Diamond p \Leftrightarrow {\sim}\Box{\sim}p; \end{array}$$

Completeness theorem

- **BK** is strongly complete wrt the class of **BK**-models

Analog of Gödel-Tarski translation

define a translation τ from the language $\mathcal{L}^{\sim} = \{\vee, \wedge, \rightarrow, \bot, \sim\}$ of the logic **N4**$^{\bot}$ to the language $\mathcal{L}^{m}$:

$$
\begin{array}{rclrcl}
\tau p & = & \Box p & \tau{\sim}p & = & {\sim}\Diamond p \\
\tau(\varphi \vee \psi) & = & \tau\varphi \vee \tau\psi & \tau{\sim}(\varphi \vee \psi) & = & \tau{\sim}\varphi \wedge \tau{\sim}\psi \\
\tau(\varphi \wedge \psi) & = & \tau\varphi \wedge \tau\psi & \tau{\sim}(\varphi \wedge \psi) & = & \tau{\sim}\varphi \vee \tau{\sim}\psi \\
\tau(\varphi \rightarrow \psi) & = & \Box(\tau\varphi \rightarrow \tau\psi) & \tau{\sim}(\varphi \rightarrow \psi) & = & \tau\varphi \wedge \tau{\sim}\psi \\
\tau\bot & = & \bot & \tau{\sim}{\sim}\varphi & = & \tau\varphi
\end{array}
$$

Theorem

τ faithfully embeds **N4**$^{\bot}$ *into* **BS4***, i.e., for any formula φ of the language $\mathcal{L}^{\sim}$,*

$$\varphi \in \mathbf{N4}^{\bot} \quad \Leftrightarrow \quad \tau\varphi \in \mathbf{BS4}.$$

Fisher Servi's approach to defining modal logics

- $\varphi \in$ **K** iff $ST_x(\varphi)$ is a classical first order tautology.
- $\varphi \in$ FS iff $ST_x(\varphi) \in$ QInt [G. Fisher Servi 1984]
- $\varphi \in$ **BK**$^{\text{FS}}$ iff $ST_x(\varphi) \in$ first order Belnap-Dunn logic

Fisher Servi's approach to defining modal logics

- $\varphi \in \mathbf{K}$ iff $ST_x(\varphi)$ is a classical first order tautology.
- $\varphi \in$ FS iff $ST_x(\varphi) \in$ QInt [G. Fisher Servi 1984]
- $\varphi \in \mathbf{BK}^{\text{FS}}$ iff $ST_x(\varphi) \in$ first order Belnap-Dunn logic

Fisher Servi's approach to defining modal logics

- $\varphi \in$ **K** iff $ST_x(\varphi)$ is a classical first order tautology.
- $\varphi \in$ FS iff $ST_x(\varphi) \in$ QInt [G. Fisher Servi 1984]
- $\varphi \in$ **BK**$^{\text{FS}}$ iff $ST_x(\varphi) \in$ first order Belnap-Dunn logic

Fisher Servi's approach to defining modal logics

- $\varphi \in$ **K** iff $ST_x(\varphi)$ is a classical first order tautology.
- $\varphi \in$ FS iff $ST_x(\varphi) \in$ QInt [G. Fisher Servi 1984]
- $\varphi \in \mathbf{BK}^{\mathrm{FS}}$ iff $ST_x(\varphi) \in$ first order Belnap-Dunn logic

First order Belnap-Dunn logic [Sano and Omori 2014]

- $\Sigma = \{R^2, P^1_1, P^1_2, \ldots, P^1_n, \ldots\}$
- $\mathfrak{M} = \langle M, \mu^\Sigma \rangle$, where

  $\mu^\Sigma(P_i) = (P_i^+, P_i^-)$ and $P_i^+, P_i^- \subseteq M$;
  $\mu^\Sigma(R) = (R^+, R^-)$ and $R^+, R^- \subseteq M^2$.

First order Belnap-Dunn logic

$$
\begin{array}{lll}
\mathfrak{M}, s \models^{+} P_i(x) & \text{iff} & s(x) \in P_i^{+}; \\
\mathfrak{M}, s \models^{-} P_i(x) & \text{iff} & s(x) \in P_i^{-}; \\
\mathfrak{M}, s \models^{+} R(x,y) & \text{iff} & (s(x), s(y)) \in R^{+}; \\
\mathfrak{M}, s \models^{-} R(x,y) & \text{iff} & (s(x), s(y)) \in R^{-}; \\
\mathfrak{M}, s \models^{+} \varphi \wedge \psi & \text{iff} & (\mathfrak{M}, s \models^{+} \varphi \text{ and } \mathfrak{M}, s \models^{+} \psi); \\
\mathfrak{M}, s \models^{-} \varphi \wedge \psi & \text{iff} & (\mathfrak{M}, s \models^{-} \varphi \text{ or } \mathfrak{M}, s \models^{-} \psi); \\
\mathfrak{M}, s \models^{+} \varphi \vee \psi & \text{iff} & (\mathfrak{M}, s \models^{+} \varphi \text{ or } \mathfrak{M}, s \models^{+} \psi); \\
\mathfrak{M}, s \models^{-} \varphi \vee \psi & \text{iff} & (\mathfrak{M}, s \models^{-} \varphi \text{ and } \mathfrak{M}, s \models^{-} \psi); \\
\mathfrak{M}, s \models^{+} \varphi \to \psi & \text{iff} & (\mathfrak{M}, s \models^{+} \varphi \Rightarrow \mathfrak{M}, s \models^{+} \psi); \\
\mathfrak{M}, s \models^{-} \varphi \to \psi & \text{iff} & (\mathfrak{M}, s \models^{+} \varphi \text{ and } \mathfrak{M}, s \models^{-} \psi); \\
\mathfrak{M}, s \not\models^{+} \bot & & \mathfrak{M}, s \models^{-} \bot
\end{array}
$$

First order Belnap-Dunn logic

$$
\begin{array}{lll}
\mathfrak{M}, s \models^{+} {\sim}\varphi & \text{iff} & \mathfrak{M}, s \models^{-} \varphi \\
\mathfrak{M}, s \models^{-} {\sim}\varphi & \text{iff} & \mathfrak{M}, s \models^{+} \varphi \\
\mathfrak{M}, s \models^{+} \forall x\varphi & \text{iff} & \forall s'(s' {\sim}^{x} s \;\Rightarrow \mathfrak{M}, s' \models^{+} \varphi) \\
\mathfrak{M}, s \models^{-} \forall x\varphi & \text{iff} & \exists s'(s' \sim^{x} s \text{ and } \mathfrak{M}, s' \models^{-} \varphi) \\
\mathfrak{M}, s \models^{+} \exists x\varphi & \text{iff} & \exists s'(s' \sim^{x} s \text{ and } \mathfrak{M}, s' \models^{+} \varphi) \\
\mathfrak{M}, s \models^{-} \exists x\varphi & \text{iff} & \forall s'(s' \sim^{x} s \;\Rightarrow \mathfrak{M}, s' \models^{-} \varphi).
\end{array}
$$

Standard translation ST_x

$$
\begin{array}{lcl}
ST_x(\bot) & = & \bot,\ p_i \in Prop; \\
ST_x(p_i) & = & P_i(x),\ p_i \in Prop; \\
ST_x(\varphi \wedge \psi) & = & ST_x(\varphi) \wedge ST_x(\psi); \\
ST_x(\varphi \vee \psi) & = & ST_x(\varphi) \vee ST_x(\psi); \\
ST_x(\varphi \to \psi) & = & ST_x(\varphi) \to ST_x(\psi); \\
ST_x({\sim}\varphi) & = & {\sim}ST_x(\varphi); \\
ST_x(\Box\varphi) & = & \forall y(R(x,y) \to ST_y(\varphi))^1; \\
ST_x(\Diamond\varphi) & = & \exists y(R(x,y) \wedge ST_y(\varphi)).
\end{array}
$$

[1] To pass from $ST_x(\varphi)$ to $ST_y(\varphi)$ we simultaneously replace all occurences of x by y and all occurences of y by x

Fisher Servi style FDE-modal logic

- $\varphi \in For(\mathcal{L}^m)$ is **BK**$^{\text{FS}}$-valid if $ST_x(\varphi)$ is a tautology of first order Belnap-Dunn logic.
- **BK**$^{\text{FS}}$-model is a **BK**-model with additional accessibility relation $\mathcal{M} = \langle W, R, R', v^+, v^- \rangle$
- Interpretation of $\Diamond$

$$\mathcal{M}, w \models^+ \Diamond\varphi \quad \text{iff} \quad \exists u(wRu \text{ and } \mathcal{M}, u \models^+ \varphi);$$
$$\mathcal{M}, w \models^- \Diamond\varphi \quad \text{iff} \quad \forall u(wR'u \Rightarrow \mathcal{M}, u \models^- \varphi).$$

- φ is **BK**$^{\text{FS}}$-valid iff φ is valid in every **BK**$^{\text{FS}}$-model.

- $\mathbf{BK}^{\mathrm{FS}}$ is the least set of formulas closed under the rules of substitution, *modus ponens*, and under the rules:

$$(RN)\ \frac{p}{\Box p},\quad (RM^{\sim\Diamond})\ \frac{\sim p \to \sim q}{\sim\Diamond p \to \sim\Diamond q}$$

- and containing the non-modal axioms of **BK** together with:

$$\Box(p \to q) \to (\Box p \to \Box q),\quad \neg{\sim}\Box p \leftrightarrow \Box\neg{\sim}p,$$

$$\neg\Diamond p \leftrightarrow \Box\neg p \text{ and } ({\sim}\Diamond p \wedge {\sim}\Diamond q) \to {\sim}\Diamond(p \vee q).$$

- $\mathbf{BK}^{\mathrm{FS}}$ is strongly complete w.r.t. the class of $\mathbf{BK}^{\mathrm{FS}}$-models.

Fisher Servi style FDE-modal logic

- $\mathbf{BK}^{\mathsf{FS}}$ and the fusion $\mathbf{BK} \otimes \mathbf{BK}$ have the same class of models.
- Are $\mathbf{BK}^{\mathsf{FS}}$ and $\mathbf{BK} \otimes \mathbf{BK}$ definitionally equivalent?

Fisher Servi style FDE-modal logic

- $\mathbf{BK}^{\mathrm{FS}}$ and the fusion $\mathbf{BK} \otimes \mathbf{BK}$ have the same class of models.
- Are $\mathbf{BK}^{\mathrm{FS}}$ and $\mathbf{BK} \otimes \mathbf{BK}$ definitionally equivalent?

Definitional equivalence of logics [Gyuris 99]

- $\mathcal{L}_1$ and $\mathcal{L}_2$ are propositional languages over *Prop*
- $\theta : For(\mathcal{L}_1) \to For(\mathcal{L}_2)$ is a structural translation if for some $\alpha : c^n \in \mathcal{L}_1 \mapsto \alpha(c)(p_1, \ldots, p_n) \in For(\mathcal{L}_2)$:

$$\theta(p) = p,\ p \in Prop;\quad \theta(c(\varphi_1, \ldots, \varphi_n)) = \alpha(c)(\theta(\varphi_1), \ldots, \theta(\varphi_n)),$$

- L_1 and L_2 are definitionally equivalent via structural translations θ and ρ if:
  1. $\Gamma \vdash_{L_1} \varphi$ implies $\theta(\Gamma) \vdash_{L_2} \theta(\varphi)$.
  2. $\Gamma \vdash_{L_2} \varphi$ implies $\rho(\Gamma) \vdash_{L_1} \rho(\varphi)$.
  3. For every $\varphi \in For(\mathcal{L}_1)$ and $\psi \in For(\mathcal{L}_2)$,

$$\varphi \Leftrightarrow [2] \rho\theta(\varphi) \in L_1 \text{ and } \psi \Leftrightarrow \theta\rho(\psi) \in L_2.$$

[2] $\Leftrightarrow$ is a Tarski congrience for **BK**

$\mathbf{BK}^{\Box}$

- $\mathbf{BK}^{\Box}$ is the $\Diamond$-free fragment of **BK**
- $\mathbf{BK}^{\Box}$ is the least set of $\mathcal{L}^{\Box}$-formulas closed under substitution, MP, and (RN) $\frac{p}{\Box p}$ and containing:
  - non-modal axioms of **BK**;
  - modal axioms:

$$\Box(p \to q) \to (\Box p \to \Box q) \qquad \text{and} \qquad \neg{\sim}\Box p \leftrightarrow \Box\neg{\sim}p.$$

- $\mathbf{BK}^{\Box}$ and **BK** are definitially equivalent.

Weakly structural translations

- $\mathcal{L}_1$, $\mathcal{L}_2$ are propositional languages over *Rrop*, $\sim \in \mathcal{L}_1 \cap \mathcal{L}_2$.
- $\theta : For(\mathcal{L}_1) \to For(\mathcal{L}_2)$ is *weakly structural* if

  $\theta \restriction \mathcal{L}_1 \setminus \{\sim\}$ is structural and for some
  $\beta : c^n \in \mathcal{L}_1 \setminus \{\sim\} \mapsto \beta(c)(p_1, q_1 \ldots, p_n, q_n) \in For(\mathcal{L}_2)$:

  $$\theta(\sim p) = \sim p,\ \ p \in Prop;$$

  $$\theta(\sim c(\varphi_1, \ldots, \varphi_n)) =$$

  $$= \beta(c)(\theta(\varphi_1), \theta(\sim\varphi_1), \ldots, \theta(\varphi_n), \theta(\sim\varphi_n)).$$

Weak definitional equivalence

L_1 and L_2 are *weakly definitionally equivalent* via weakly structural translations θ and ρ if the following conditions hold.

1. For $\Gamma \cup \{\varphi\} \subseteq For(\mathcal{L}_1)$, $\Gamma \vdash_{L_1} \varphi$ implies $\theta(\Gamma) \vdash_{L_2} \theta(\varphi)$.
2. For $\Gamma \cup \{\varphi\} \subseteq For(\mathcal{L}_2)$, $\Gamma \vdash_{L_2} \varphi$ implies $\rho(\Gamma) \vdash_{L_1} \rho(\varphi)$.
3. For every $\varphi \in For(\mathcal{L}_1)$ and $\psi \in For(\mathcal{L}_2)$,

$$\varphi \leftrightarrow \rho\theta(\varphi) \in L_1 \text{ and } \psi \leftrightarrow \theta\rho(\psi) \in L_2.$$

$\theta : For(\mathcal{L}^{\Box}) \to For(\mathcal{L}^{\Box\blacksquare})$

- θ preserves propositional variables and constant $\bot$, commutes with connectives $\vee$, $\wedge$, $\to$, $\Box$, and

$$\theta(\Diamond\varphi) = {\sim}\Box{\sim}\theta(\varphi).$$

- For strongly negated formulas:

$$\theta({\sim}p) = {\sim}p, \;\; \theta({\sim}\bot) = {\sim}\bot, \;\; \theta({\sim}(\varphi \vee \psi)) = \theta({\sim}\varphi) \wedge \theta({\sim}\psi),$$

$$\theta({\sim}(\varphi \wedge \psi)) = \theta({\sim}\varphi) \vee \theta({\sim}\psi), \;\; \theta({\sim}(\varphi \to \psi)) = \theta(\varphi) \wedge \theta({\sim}\psi),$$

$$\theta({\sim}\Box\varphi) = {\sim}\Box{\sim}\theta({\sim}\varphi), \;\; \theta({\sim}\Diamond\varphi) = \blacksquare\theta({\sim}\varphi).$$

$\rho : For(\mathcal{L}^{\Box\blacksquare}) \to For(\mathcal{L}^{\Box})$

- ρ also preserves propositional variables and constant $\bot$ and commutes with connectives $\vee$, $\wedge$, $\to$, $\Box$.
- For strongly negated formulas:

$$\rho({\sim}p) = {\sim}p, \;\; \rho({\sim}\bot) = {\sim}\bot, \;\; \rho({\sim}(\varphi \vee \psi)) = \rho({\sim}\varphi) \wedge \rho({\sim}\psi),$$

$$\rho({\sim}(\varphi \wedge \psi)) = \rho({\sim}\varphi) \vee \rho({\sim}\psi), \;\; \rho({\sim}(\varphi \to \psi)) = \rho(\varphi) \wedge \rho({\sim}\psi),$$

$$\rho({\sim}\Box\varphi) = {\sim}\Box{\sim}\rho({\sim}\varphi), \;\; \rho({\sim}\blacksquare\varphi) = \neg{\sim}\Diamond{\sim}\neg\rho({\sim}\varphi).$$

- $\mathbf{BK}^{\mathrm{FS}}$ and $\mathbf{BK}^{\Box} \otimes \mathbf{BK}^{\Box}$ are weakly definitionally equivalent via θ and ρ

KN4 [Goble, 2006] based on BN4

- $\mathcal{L}^{\Box}_{\Rightarrow} := \{\vee, \wedge, \Rightarrow, \sim, \Box\}$.
- $\mathrm{BK}^{\Box}$-models
- $$\begin{array}{lll} \mathcal{M}, w \models^{+} \varphi \Rightarrow \psi & \text{iff} & ((\mathcal{M}, w \models^{+} \varphi \text{ implies } \mathcal{M}, w \models^{+} \psi) \text{ and} \\ & & (\mathcal{M}, w \models^{-} \psi \text{ implies } \mathcal{M}, w \models^{-} \varphi)) \\ \mathcal{M}, w \models^{-} \varphi \Rightarrow \psi & \text{iff} & (\mathcal{M}, w \models^{+} \varphi \text{ and } \mathcal{M}, w \models^{-} \psi). \end{array}$$

HKN4, Hilbert style calculus for KN4

- Non-modal axioms

$\varphi \Rightarrow \varphi$
$(\varphi \wedge \psi) \Rightarrow \varphi, \quad (\varphi \wedge \psi) \Rightarrow \psi$
$((\varphi \Rightarrow \psi) \wedge (\varphi \Rightarrow \chi)) \Rightarrow (\varphi \Rightarrow (\psi \wedge \chi))$
$\varphi \Rightarrow (\varphi \vee \psi), \quad \psi \Rightarrow (\varphi \vee \psi)$
$((\varphi \Rightarrow \chi) \wedge (\psi \Rightarrow \chi)) \Rightarrow ((\varphi \vee \psi) \Rightarrow \chi)$
$(\varphi \wedge (\psi \vee \chi)) \Rightarrow ((\varphi \wedge \psi) \vee (\varphi \wedge \chi))$
$(\varphi \Rightarrow {\sim}\psi) \Rightarrow (\psi \Rightarrow {\sim}\varphi)$
${\sim}{\sim}\varphi \Rightarrow \varphi$
$({\sim}\varphi \wedge \psi) \Rightarrow (\varphi \Rightarrow \psi)$
${\sim}\varphi \Rightarrow (\varphi \vee (\varphi \Rightarrow \psi))$
$\varphi \vee ({\sim}\psi \vee (\varphi \Rightarrow \psi))$
$\varphi \Rightarrow ((\varphi \Rightarrow {\sim}\varphi) \Rightarrow {\sim}\varphi)$
$\varphi \vee ({\sim}\varphi \Rightarrow (\varphi \Rightarrow \psi))$

- Modal axioms

K) $\Box(\varphi \Rightarrow \psi) \Rightarrow (\Box\varphi \Rightarrow \Box\psi)$
C) $(\Box\varphi \wedge \Box\psi) \Rightarrow \Box(\varphi \wedge \psi)$
Bel) $\Box(\varphi \vee \psi) \Rightarrow ({\sim}\Box{\sim}\varphi \vee \Box\psi)$
Nec) If φ is an axiom then so is $\Box\varphi$.

HKN4, rules

- Adj) $\varphi \quad \psi \;/\; \varphi \wedge \psi$
  MP) $\varphi \quad \varphi \Rightarrow \psi \;/\; \psi$
  Prefix) $\varphi \Rightarrow \psi \;/\; (\chi \Rightarrow \varphi) \Rightarrow (\chi \Rightarrow \psi)$
  Suffix) $\varphi \Rightarrow \psi \;/\; (\psi \Rightarrow \chi) \Rightarrow (\varphi \Rightarrow \chi)$
- an infinite set *XMP* of extended *modus ponens* rules

HKN4, extended *modus ponens* rules

- *MP**, $\varphi \wedge (\varphi \Rightarrow \psi) \ / \ (\varphi \wedge (\varphi \Rightarrow \psi)) \wedge \psi$, is in *XMP*
- If a rule *r* is in *XMP*, then so are all instances of *Cr*, *Dr*, *Nr* and *Mr*.
- If $r = \varphi \ / \ \psi$, then:

Dr	$\chi \vee \varphi \ / \ \chi \vee \psi$
Cr	$\chi \wedge \varphi \ / \ \chi \wedge \psi$
Nr	$\Box\varphi \ / \ \Box\psi$
Mr	$\Diamond\varphi \ / \ \Diamond\psi$

Tableau for BK$^{\Box-}$

$$\begin{array}{c}\varphi \wedge \psi, +i\\ \downarrow\\ \varphi, +i\\ \psi, +i\end{array} \qquad \begin{array}{c}\varphi \wedge \psi, -i\\ \swarrow \quad \searrow\\ \varphi, -i \quad \psi, -i\end{array} \qquad \begin{array}{c}\varphi \vee \psi, +i\\ \swarrow \quad \searrow\\ \varphi, +i \quad \psi, +i\end{array} \qquad \begin{array}{c}\varphi \vee \psi, -i\\ \downarrow\\ \varphi, -i\\ \psi, -i\end{array} \qquad \begin{array}{c}\varphi \to \psi, +i\\ \swarrow \quad \searrow\\ \varphi, -i \quad \psi, +i\end{array} \qquad \begin{array}{c}\varphi \to \psi, -\\ \downarrow\\ \varphi, +i\\ \psi, -i\end{array}$$

$$\begin{array}{c}\sim\sim\varphi, \underline{+}i\\ \downarrow\\ \varphi, \underline{+}i\end{array} \qquad \begin{array}{c}\sim(\varphi \wedge \psi), \underline{+}i\\ \downarrow\\ \sim\varphi \vee \sim\psi, \underline{+}i\end{array} \qquad \begin{array}{c}\sim(\varphi \vee \psi), \underline{+}i\\ \downarrow\\ \sim\varphi \wedge \sim\psi, \underline{+}i\end{array} \qquad \begin{array}{c}\sim(\varphi \to \psi), \underline{+}i\\ \downarrow\\ \varphi \wedge \sim\psi, \underline{+}i\end{array}$$

$$\begin{array}{c}\Box\varphi, +i\\ irj\\ \downarrow\\ \varphi, +j\end{array} \qquad \begin{array}{c}\Box\varphi, -i\\ \downarrow\\ irj\\ \varphi, -j\end{array} \qquad \begin{array}{c}\sim\Box\varphi, +i\\ \downarrow\\ irj\\ \sim\varphi, +j\end{array} \qquad \begin{array}{c}\sim\Box\varphi, -i\\ irj\\ \downarrow\\ \sim\varphi, -j\end{array}$$

Tableau for KN4

$$
\begin{array}{cccc}
\multicolumn{4}{c}{\varphi \Rightarrow \psi, +i} \\
\swarrow & \downarrow & \downarrow & \searrow \\
\varphi, -i & \varphi, -i & \psi, +i & \psi, +i \\
\sim\psi, -i & \sim\varphi, +i & \sim\psi, -i & \sim\varphi, +i
\end{array}
\qquad
\begin{array}{cc}
\multicolumn{2}{c}{\varphi \Rightarrow \psi, -i} \\
\swarrow & \searrow \\
\varphi, +i & \sim\psi + i \\
\psi, -i & \sim\varphi - i
\end{array}
$$

$$
\begin{array}{c}
\sim(\varphi \Rightarrow \psi), +i \\
\downarrow \\
\varphi, +i \\
\sim\psi, -i
\end{array}
\qquad
\begin{array}{c}
\sim(\varphi \Rightarrow \psi), -i \\
\downarrow \\
\varphi, +i \\
\sim\psi, -i \\
\psi, -i \\
\sim\varphi, -i
\end{array}
$$

Definitional equivalence

- $\mathrm{BK}^{\Box-}$ is a $\bot$-free fragment of $\mathbf{BK}^{\Box}$.
- γ: $\mathit{For}(\mathcal{L}^{\Box}_{\Rightarrow}) \longrightarrow \mathit{For}(\mathcal{L}^{\Box-})$ preserves propositional variables, commutes with $\sim \Box$, $\wedge$, $\vee$, and $\gamma(\varphi \Rightarrow \psi) = (\gamma(\varphi) \to \gamma(\psi)) \wedge (\gamma({\sim}\psi) \to \gamma({\sim}\varphi))$.
- δ: $\mathit{For}(\mathcal{L}^{\Box-}) \longrightarrow \mathit{For}(\mathcal{L}^{\Box}_{\Rightarrow})$ preserves propositional variables, commutes with $\sim \Box$, $\wedge$, $\vee$, and $\delta(\varphi \to \psi) = (\delta(\varphi) \Rightarrow (\delta(\varphi) \Rightarrow \delta(\psi))) \vee \delta(\psi)$.
- KN4 and $\mathrm{BK}^{\Box-}$ are definitionally equivalent via γ and δ.

MBL, Modal Bilattice Logic [Jung, Rivieccio 2012]

- $\mathcal{L}^{\mathsf{MBL}} = \{\wedge, \vee, \otimes, \oplus, \rightarrow, \sim, \Box, \bot, \top, \mathsf{b}, \mathsf{n}\}$.
- In case of BK,

$$V(\Box\varphi, w) = \inf_{\leq_t}\{V(\varphi, u) \mid wRu\}$$

- In case of MBL, both V and R are four-valued and

$$V(\Box\varphi, w) = \inf_{\leq_t}\{wRu \Rightarrow V(\varphi, u) \mid u \in W\},$$

where $\varphi \Rightarrow \psi := (\varphi \rightarrow \psi) \wedge (\sim\psi \rightarrow \sim\varphi)$

MBL-validities

- For MBL modality

$$
\begin{array}{lll}
\mathcal{M}, w \models^{+} \Box\varphi & \text{iff} & \forall u(wR_{+}u \text{ implies } \mathcal{M}, u \models^{+} \varphi) \text{ and} \\
 & & \forall u(wR_{-}u \text{ implies } \mathcal{M}, u \not\models^{-} \varphi); \\
\mathcal{M}, w \models^{-} \Box\varphi & \text{iff} & \exists u(wR_{+}u \text{ and } \mathcal{M}, u \models^{-} \varphi).
\end{array}
$$

Embedding MBL^- into $\mathbf{BK}^\Box \otimes \mathbf{BK}^\Box$

$\zeta : For(\mathcal{L}^\Box) \to For(\mathcal{L}^{\Box\blacksquare})$ preserves propositional variables and constants, commutes with the connectives $\wedge$, $\vee$, $\to$, and :

$$\zeta({\sim}p) = {\sim}p, \quad \zeta({\sim}(\varphi \vee \psi)) = \zeta({\sim}\varphi) \wedge \zeta({\sim}\psi),$$

$$\zeta({\sim}(\varphi \wedge \psi)) = \zeta({\sim}\varphi) \vee \zeta({\sim}\psi), \quad \zeta({\sim}(\varphi \to \psi)) = \zeta(\varphi) \wedge \gamma({\sim}\psi),$$

$$\zeta(\Box\varphi) = \Box\zeta(\varphi) \wedge \blacksquare\zeta(\neg{\sim}\varphi), \qquad \zeta({\sim}\Box\varphi) = {\sim}\Box{\sim}\zeta({\sim}\varphi).$$

Theorem

Let $\Gamma \cup \{\chi\} \subseteq For(\mathcal{L}^{\mathrm{MBL}})$. $\Gamma \models_{\mathrm{MBL}^-} \chi$ *iff* $\zeta(\Gamma) \models_{\mathrm{BK}^\Box \times \mathrm{BK}^\Box} \zeta(\chi)$.

Thank You!