Solovay's Completeness without Fixed Points

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Solovay's Theorem

Theorem (Solovay 1976)

For a modal formula φ, we have $\mathrm{GL} \vdash \varphi$ *iff for every arithmetical interpretation f we have* $\mathrm{PA} \vdash f(\varphi)$*.*

General proof plan:
"Soundness part" follows from Löb's theorem.
In order to prove "completeness part", assuming $\mathrm{GL} \nvdash \varphi$, we need to give an arithmetical interpretation f such that $\mathrm{PA} \nvdash f(\varphi)$.
The latter is achieved by taking a GL-frame $(W, \prec) \not\models \varphi$ and giving an assignment of arithmetical sentences F_w to worlds $w \in W$.
The assignment should give an embedding of the Magari algebra of the frame into the Magari algebra of $\mathrm{PA} + F$, for some F.

Parity Based Natural Orey Sentence for PA

Let us consider sentence

H: "$\neg\mathrm{Con}(\mathrm{PA})$ and the least n such that $\neg\mathrm{Con}(\mathrm{I}\Sigma_n)$ is odd"

Let us prove that

$$\mathrm{PA} \rhd \mathrm{PA} + \mathrm{H} \text{ and } \mathrm{PA} \rhd \mathrm{PA} + \neg\mathrm{H}$$

By Orey-Hajek Theorem it is enough to show that

$$\mathrm{PA} \vdash \mathrm{Con}(\mathrm{I}\Sigma_k + \mathrm{H}) \wedge \mathrm{Con}(\mathrm{I}\Sigma_k + \neg\mathrm{H}), \text{ for each } k$$

Indeed, we derive in PA:

$$\mathrm{Con}(\mathrm{I}\Sigma_{k+1}) \wedge \mathrm{Con}(\mathrm{I}\Sigma_k)$$
$$\mathrm{Con}(\mathrm{I}\Sigma_{k+1} + \neg\mathrm{Con}(\mathrm{I}\Sigma_{k+1}))) \wedge \mathrm{Con}(\mathrm{I}\Sigma_k + \neg\mathrm{Con}(\mathrm{I}\Sigma_k)))$$
$$\mathrm{Con}(\mathrm{I}\Sigma_k + \mathrm{Con}(\mathrm{I}\Sigma_k) + \neg\mathrm{Con}(\mathrm{I}\Sigma_{k+1}))) \wedge \mathrm{Con}(\mathrm{I}\Sigma_k + \neg\mathrm{Con}(\mathrm{I}\Sigma_k)))$$
$$\mathrm{Con}(\mathrm{I}\Sigma_k + \mathrm{H}) \wedge \mathrm{Con}(\mathrm{I}\Sigma_k + \neg\mathrm{H})$$

No Jump Sentence

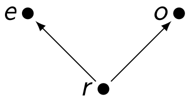
Super-exponential function and logarithm:
$\exp^\star(0) = 1, \quad \exp^\star(n+1) = 2^{\exp^\star(n)}$
$\log^\star(n) = k$, where k is the least number such that $\exp^\star(k) \geq n$.

$F_o \leftrightharpoons \exists p((\log^\star(p) \equiv 1 \pmod 2)) \wedge p{:}\Box\bot \wedge \forall p' < p(\neg p'{:}\Box\bot));$
$F_e \leftrightharpoons \exists p((\log^\star(p) \equiv 0 \pmod 2)) \wedge p{:}\Box\bot \wedge \forall p' < p(\neg p'{:}\Box\bot)).$

We will show that

$$\mathrm{PA} \vdash \Diamond F_o \leftrightarrow \Diamond F_e \leftrightarrow \Diamond\neg F_o \leftrightarrow \Diamond\neg F_e \leftrightarrow \Diamond\top.$$

With $F_r \leftrightharpoons \Diamond\top$ it will give an embedding in arithmetic of

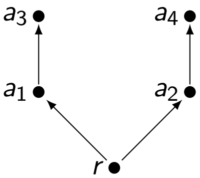


Double Jump Sentence

$$C_{a_1} \leftrightharpoons \exists p((\log^\star(p) \equiv 1 \pmod 2) \land p{:}\Box\Box\bot \land \forall p' < p(\neg p'{:}\Box\Box\bot));$$
$$C_{a_2} \leftrightharpoons \exists p((\log^\star(p) \equiv 0 \pmod 2) \land p{:}\Box\Box\bot \land \forall p' < p(\neg p'{:}\Box\Box\bot));$$
$$F_{a_1} \leftrightharpoons \Diamond\top \land C_{a_1};$$
$$F_{a_2} \leftrightharpoons \Diamond\top \land C_{a_2};$$
$$F_{a_3} \leftrightharpoons \Box\bot \land C_{a_1};$$
$$F_{a_4} \leftrightharpoons \Box\bot \land C_{a_2};$$
$$F_r \leftrightharpoons \Diamond\Diamond\top.$$

We have

$$\mathrm{PA} \nvdash \Diamond\top \to \Diamond F_{a_3} \text{ and } \mathrm{PA} \nvdash \Diamond\top \to \Diamond\neg F_{a_3}.$$

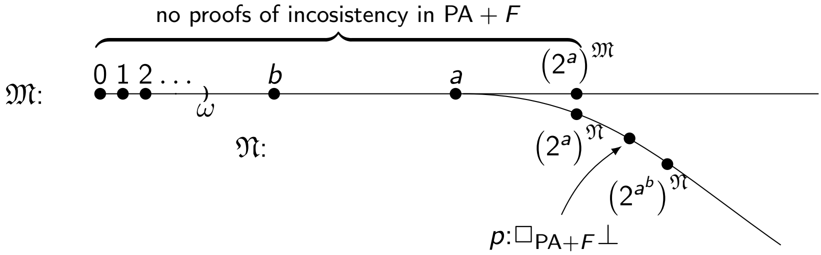


Injecting Inconsistencies in Models of Arithmetic

Theorem (P. Hajek, R. Solovay, P. Pudlak, J. Krajíček, A. Visser, and R. Verbrugge)

Suppose $\mathfrak{M}$ *is a model of* $\mathsf{PA} + F$, $a, b \in \mathfrak{M}$ *are non-standard numbers,* $\mathfrak{M} \models b < a$, *and* $\mathfrak{M} \models \forall p(p{:}\Box_{\mathsf{PA}+F}\bot \to p > 2^a)$. *Then there exists a model* $\mathfrak{N}$ *such that:*

1. $\mathfrak{M} \restriction a = \mathfrak{N} \restriction a$;
2. $\mathfrak{N} \models \forall p(p{:}\Box_{\mathsf{PA}+F}\bot \to p > 2^a)$;
3. $\mathfrak{N} \models \exists p(2^a < p \leq 2^{a^b} \wedge p{:}\Box_{\mathsf{PA}+F}\bot)$.



Model Theory in PA

We will need to formalize results obtained by model-theoretic methods in PA.
In order to achieve this we will use conservative extension that is more suitable for formalization of model theory.

Within ACA_0 it is possible to formalize considerable part of "normal" formalization of model theory (for more details see S. Simpson book "Subsystems of Second Order Arithmetic").

The key model-theoretic result that is used in the proof of H-S-P-K-V-V is

Theorem (Omitting Types Theorem)

Suppose T *is a consistent theory that locally omits the set of formulas* $\Sigma(x_1, \ldots, x_n)$*. Then there is a model* $\mathfrak{M}$ *of* T *that omits the set* Σ*.*

Omitting Types theorem is provable in ACA_0.

Question

What is reverse mathematics strength of Omitting Types theorem?

Parity of $\log^\star$ for the Least Inconsistency

$$F_o \leftrightharpoons \exists p((\log^\star(p) \equiv 1 \pmod 2) \land p{:}\Box\bot \land \forall p' < p(\neg p'{:}\Box\bot));$$
$$F_e \leftrightharpoons \exists p((\log^\star(p) \equiv 0 \pmod 2) \land p{:}\Box\bot \land \forall p' < p(\neg p'{:}\Box\bot)).$$

Let us prove that $\mathrm{PA} \vdash \Diamond F_o \leftrightarrow \Diamond F_e \leftrightarrow \Diamond\top$.

Since there is no essential difference between F_o and F_e, it will be enough to show that $\mathrm{PA} \vdash \Diamond\top \to \Diamond F_e$.

Let us reason in ACA_0. Assume $\Diamond\top$. By Gödel's Second Incompleteness theorem, we have $\Diamond\Box\bot$. Thus we have a model $\mathfrak{M}$ of $\mathrm{PA} + \Box\bot$. Since $\Diamond\top$, if $\mathfrak{M} \models p{:}\Box\bot$ for some $p \in \mathfrak{M}$, then p is non-standard. $\mathfrak{M}$ is a model of PA, hence there is the least $p_0 \in \mathfrak{M}$ such that $\mathfrak{M} \models p_0{:}\Box\bot$. The number $\log^\star(p_0)$ is non-standard, since $\mathfrak{M} \models p_0 \le \exp^\star(\log^\star(p_0))$ and $\exp^\star(\log^\star(p_0))$ would be standard if $\log^\star(p_0)$ is standard. We put $b = \log^\star(p_0) - 2$ if $\log^\star(p_0)$ is even and $b = \log^\star(p_0) - 3$ if $\log^\star(p_0)$ is odd. We put $a = \exp^\star(b)$.

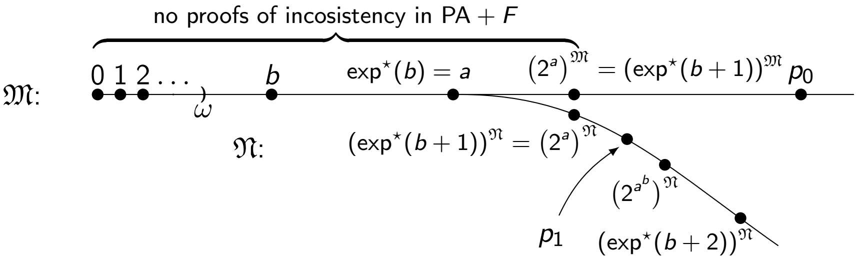
Parity of $\log^\star$ for the Least Inconsistency

We fork the model $\mathfrak{M}$ and obtain a model $\mathfrak{N}$ such that $\mathfrak{N} \models 2^a < p_1 < 2^{a^b}$, where $p_1 \in \mathfrak{N}$ is the least $p_1{:}\Box\bot$ in $\mathfrak{N}$.

It is easy to see that $\mathfrak{N} \models a^b < 2^a$, hence

$$\mathfrak{N} \models \exp^\star(b+1) = 2^a < p_1 < 2^{a^b} < 2^{2^a} = \exp^\star(b+2).$$

Thus $\mathfrak{N} \models \log^\star(p_1) = b+2$ and $\mathfrak{N} \models \log^\star(p_1) \equiv 0 \pmod 2)$.



Therefore $\mathfrak{N} \models F_e$. Hence $\Diamond F_e$.

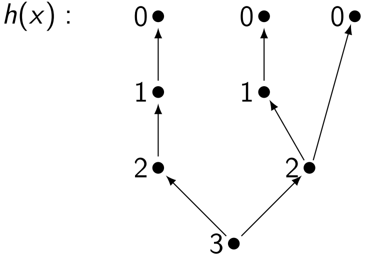
Thus PA $\vdash \Diamond\top \to \Diamond F_e$.

Kripke Frames

Suppose $(W, \prec)$ is a tree-like frame with the root r.

The height of a world $a \in W$:

$$h(a) = \sup(\{0\} \cup \{h(b) + 1 \mid a \prec b\}).$$



Embedding Kripke Frames by Parity of $\log^\star$: Simple Case

Suppose frame $(W, \prec)$ is such that all the paths from root to leafs are of the same length.

Then for every $a \in W$, all the immediate successors $b_0, \ldots, b_{n-1} \in W$ (i.e. all $b \in W$ such that $a \prec b$ and for all $c \in W$ we don't have $a \prec c \prec b$) have the same height:

$$h(a) - 1 = h(b_0) = \ldots = h(b_{n-1}).$$

Suppose for a world $a \in W$ the worlds $b_0, \ldots, b_{n-1} \in W$ are all the immediate successors of a. We put

$$C_{b_i} \leftrightharpoons \exists p((\log^\star(p) \equiv i \pmod{n}) \wedge p{:}\Box^{h(a)}\bot \wedge \forall p'(\neg p'{:}\Box^{h(a)}\bot)).$$

We define

1. $F_r \leftrightharpoons \Diamond^{h(r)}\top$;
2. $F_a \leftrightharpoons \Diamond^{h(a)}\top \wedge \bigwedge\limits_{r \prec b \preceq a} C_b$.

Embedding Kripke Frames by Parity of $\log^\star$

Again, suppose $(W, \prec)$ is a tree-like frame with the root r. We have no additional restrictions.

The only difference will be the definitions of C_a.

Suppose for a world $a \in W$ the worlds $b_0, \ldots, b_{n-1} \in W$ are all the immediate successors and $h(b_{n-1}) = h(a) - 1$. For $i < n-1$, we put

$$C_{b_i} \leftrightharpoons \exists p((\log^\star(p) \equiv i \pmod{n})) \wedge p{:}\Box^{h(a)}\bot \wedge \forall p'(\neg p'{:}\Box^{h(a)}\bot \wedge \exists u < 2^{2^p}(u{:}\Box^{h(b_i)+1}\bot)).$$

We put

$$C_{b_{n-1}} \leftrightharpoons \Box^{h(a)}\bot \wedge \neg \bigwedge_{i<n-1} C_{b_i}.$$

Again, we could prove that F_a will give an embedding of $(W, \prec)$ in arithmetic.

Thank You!