Systems of propositions referring to each other: a model-theoretic view

Denis I. Saveliev

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Theory Ξ

An important type of paradoxes combines negation and a kind of self-reference. The classical Liar paradox is a paradigmatic instance.

A similar paradox containing no (explicit) self-reference was proposed by Yablo about twenty years ago. This new paradox can be considered as an unfolding of the Liar paradox: it consists of propositions indexed by natural numbers such that each of the propositions states "all propositions with greater indices are wrong".

Remark. The Yablo paradox generated plenty of papers (by Priest, Sorensen, Beall, Forster, and others) mainly discussed whether the paradox uses a kind of an implicit self-reference. This disscusion is out of the subject of the talk.

Our purpose:

(a) to investigate arbitrary systems of propositions some of them state that some of them (possibly, themselves) are wrong,
(b) to establich which of these systems are paradoxical and which are not.

For this, we introduce a first-order theory Ξ of a language with one unary and one binary predicates, T and U, consisting of two easy axioms, $A1$ and $A2$, which express natural relationships of propositions and their truth values.

Definition. Theory Ξ:

$A1 \qquad \forall xy\,(Tx \to (Uxy \to \neg\, Ty)),$

$A2 \qquad \forall x\,(\neg\, Tx \wedge \exists y\, Uxy \to \exists y\,(Uxy \wedge Ty)).$

Heuristically:
variables x, y mean propositions,
Tx means "x is true",
Uxy means "x states that y is wrong".

Thus:
$A1$ means "if x is true and states that y is wrong, then this y is indeed wrong", while
$A2$ means "if x is wrong and states that some y are wrong, then some of these y's witnesses that x is wrong".

Note that the interpretation allows mute but wrong x's.

What are properties of Ξ?

Theorem. Ξ *is a* Π^0_2 *but not a* Σ^0_2 *theory.*

Sketch of proof.

1. Axiom $A1$ is equivalent to the Π^0_1 sentence

$$\forall xy\, (\neg\, Tx \vee \neg\, Uxy \vee \neg\, Ty)$$

and axiom $A2$ is equivalent to the Π^0_2 sentence

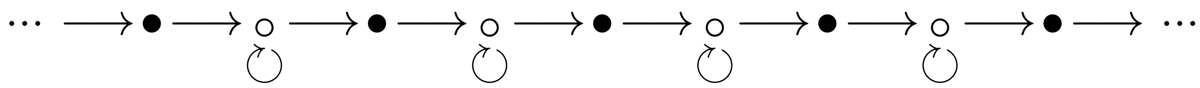
$$\forall xy\, \exists z\, (Tx \vee \neg\, Uxy \vee (Tz \wedge Uxz)).$$

Hence, Π^0_2.

Why not Σ^0_2?

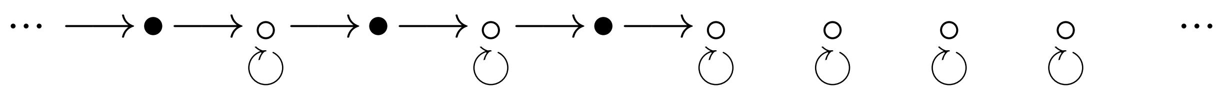
2. Let $\mathfrak{A} = (X, T, U)$ where:
$X = \mathbb{Z}$ (the set of integers),
$T = \{2n + 1 : n \in \mathbb{Z}\}$,
$U = \{(n, n + 1), (2n, 2n) : n \in \mathbb{Z}\}$.

We can picture it as follows:



(White nodes = elements x with $\neg Tx$,
black nodes = elements x with Tx,
arrows = pairs of elements x, y with Uxy.)

For any $k \in \mathbb{Z}$, let $\mathfrak{A}_k$ be the submodel of $\mathfrak{A}$ with the universe $\{n \leq 2k : n \in \mathbb{Z}\} \cup \{2n : n \in \mathbb{Z}\}$:



Facts:
1. $\mathfrak{A}$ satisfies Ξ.
2. $\mathfrak{A}_k$ do not satisfy Ξ and are isomorphic.

Toward a contradiction, assume that Ξ can be axiomatized by a set Γ of Σ^0_2 sentences. So $\mathfrak{A} \models \Gamma$.

Claim. For any $\gamma \in \Gamma$ there is $k \in \mathbb{Z}$ with $\mathfrak{A}_k \models \gamma$.

[Why? Any sentence $\gamma \in \Gamma$ has the form

$$\exists x_0 \dots x_m \forall y_0 \dots y_n \, \varphi(x_0, \dots, x_m, y_0, \dots, y_n)$$

with φ open. Since $\mathfrak{A} \models \gamma$, there are $a_0, \dots, a_m \in X$ such that

$$\mathfrak{A} \models \forall y_0 \dots y_n \, \varphi(a_0, \dots, a_m, y_0, \dots, y_n).$$

Submodels preserve universal formulas, so any submodel of $\mathfrak{A}$ satisfies the same whenever it has the $a_0, \dots, a_m$. Hence, if $k = \max_{i \leq m} a_i$ then

$$\mathfrak{A}_k \models \varphi(a_0, \dots, a_m, y_0, \dots, y_n),$$

and so $\mathfrak{A}_k \models \gamma$.]

Now: all the $\mathfrak{A}_k$ are isomorphic and so satisfy the same sentences. Therefore, each $\mathfrak{A}_k$ should satisfy every $\gamma \in \Gamma$, and thus Ξ. A contradiction.

Hence, not Σ^0_2.

Other properties of Ξ:

Proposition. *The theory Ξ:*
has atomic saturated models of any cardinality,
does not admit quantifier elimination,
is neither complete nor model complete,
not Horn,
not categorical in any cardinality,
undecidable.

[E.g. why not categorical in cardinality 1? There are single-point models, one with Tx and another with $\neg Tx$:

The same for any cardinality.]

Let $M(\Gamma)$ be the class of models of a theory Γ.

Model-theoretic operations which preserve or do not preserve Ξ:

Proposition. $M(\Xi)$ *is closed under:*
upper cones w.r.t. U*,*
disjoint sums and summands,
unions of increasing chains,
dualization,
ultrafilter extensions,
1*-sandwichs.*

Proposition. $M(\Xi)$ *is not closed under:*
submodels,
direct products and factors,
bisimulations,
images and pre-images of strong homomorphisms,
non-empty intersections of two elementary submodels,
non-empty intersections of decreasing chains,
2*-sandwichs.*

Collapses of models of Ξ

We describe a procedure that will be used for a natural classification of of models of Ξ.

Terminology.
Let (X,T,U) and (X',T',U') be models of the language of Ξ. A map $\pi : X \to X'$ is:
a *quasi-isomorphism* w.r.t. T iff

$$\forall x\ (Tx\ \leftrightarrow\ T'\pi x),$$

a *bounded morphism* w.r.t. U iff

$$\forall xy\ (Uxy\ \to\ U'\pi x\pi y),$$
$$\forall xy'\,\exists y\ (U'\pi xy'\ \to\ \pi y = y' \wedge Uxy).$$

Note that bounded morphisms (aka p-morphisms) are stronger than strong homomorphisms.

Lemma. *Let* (X, T, U) *and* (X', T', U') *be such that there exists* $\pi : X \to X'$ *which is:*
a surjection,
a quasi-isomorphism w.r.t. T*,*
a bounded morphism w.r.t. U*.*
Then $(X, T, U) \models \Xi$ *iff* $(X', T', U') \models \Xi$*.*

Informally: π "glues together" some of points with the same value of T.

After glueing *all* such points of a given model, it collapses to a model consisting of at most two points.

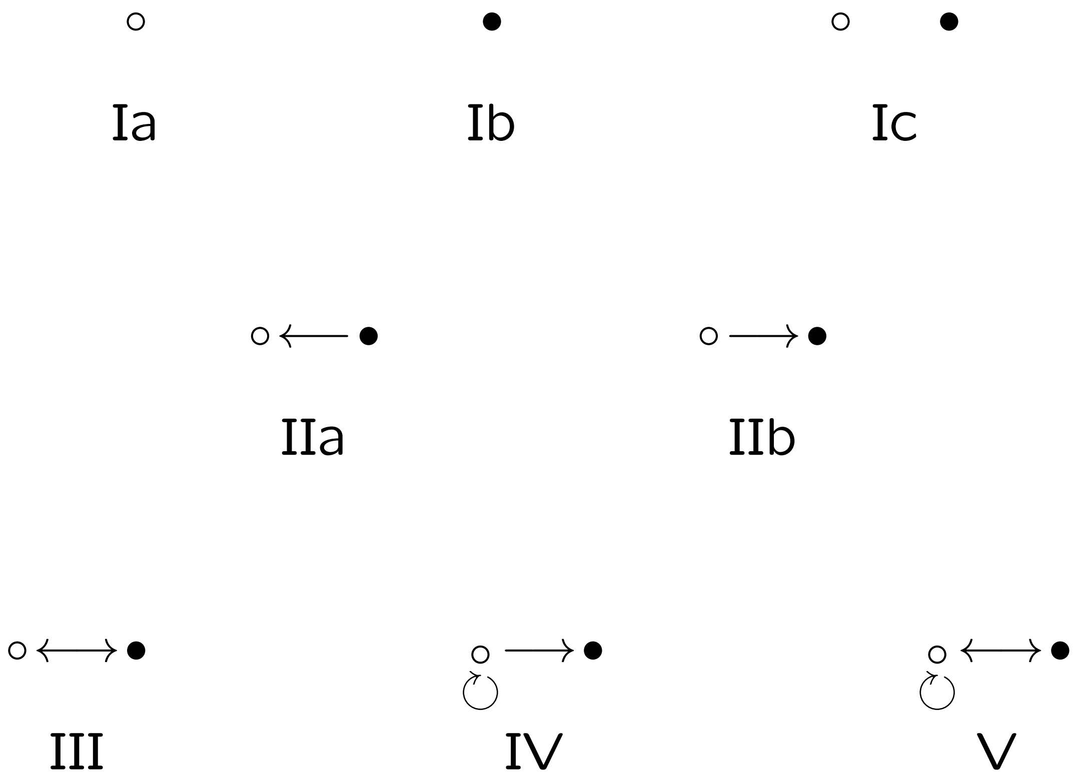
Definition. The *collapse* of a model (X, T, U) is the model (X', T', U') with the map $\pi : X \to \{0, 1\}$ defined as follows:
$\pi x = 1$ if Tx, and $\pi x = 0$ otherwise,
$X' = \{\pi x : x \in X\}$,
$T'x'$ iff $x' = 1$ (i.e. $T'1 \wedge \neg T'0$),
$U'x'y'$ iff $\exists xy \, (\pi x = x' \wedge \pi y = y' \wedge Uxy)$.

Theorem. *A model satisfies Ξ iff its collapse does. There exist exactly eight models that are the collapses of models of Ξ, listed below:*

	X	T	U
Ia	$\{0\}$	$\emptyset$	$\emptyset$
Ib	$\{1\}$	$\{1\}$	$\emptyset$
Ic	$\{0,1\}$	$\{1\}$	$\emptyset$
IIa	$\{0,1\}$	$\{1\}$	$\{(0,1)\}$
IIb	$\{0,1\}$	$\{1\}$	$\{(1,0)\}$
III	$\{0,1\}$	$\{1\}$	$\{(0,1),(1,0)\}$
IV	$\{0,1\}$	$\{1\}$	$\{(0,0),(0,1)\}$
V	$\{0,1\}$	$\{1\}$	$\{(0,0),(0,1),(1,0)\}$

(Models with isomorphic U are denoted by the same Roman letters.)

The picture of the 8 models:



Examples.
1. Any model with empty U collapses to one of Ia, Ib, Ic.
2. The model $\mathfrak{A}$ from the proof above (about complexity of Ξ) collapses to V.

Classification of models of Ξ

The class $M(\Xi)$ is divided into eight subclasses consisting of the pre-images of eight models above under the collapse.

For $X \in \{\text{Ia, Ib, Ic, IIa, IIb, III, IV, V}\}$, let $\Xi(X)$ denote the theory of pre-images of X.

Refining the theorem above:

Theorem.
1. $\Xi(\text{Ia})$, $\Xi(\text{Ib})$ *are* Π^0_1 *but not* Σ^0_1 *theories.*
2. $\Xi(\text{Ic})$, $\Xi(\text{IIa})$, $\Xi(\text{IIb})$, $\Xi(\text{III})$, $\Xi(\text{IV})$ *are* Δ^0_2 *but neither* Π^0_1 *nor* Σ^0_1 *theories.*
3. $\Xi(\text{V})$ *is a* Π^0_2 *but not* Σ^0_2 *theory.*

Paradoxicality

Definition. A model (X, U) is *non-paradoxical* iff it can be expanded to some model (X, T, U) of Ξ, and *paradoxical* otherwise.

Examples.
1. A model of the Liar Paradox consists of one reflexive point:

and it is paradoxical.
2. A model of the Yablo Paradox is isomorphic to natural numbers with their usual ordering, and it is paradoxical.
3. A cycle is paradoxical iff its length is odd. (E.g. the Liar is a cycle of length 1.)

More examples can be obtained by considering specific classes of relations.

Specific relations

Proposition. *If U is reflexive, then (X, U) is paradoxical.*

This generalizes the Liar Paradox, the model of which consists of one reflexive point.

Given (X, U), a subset C of X is *U-cofinal* in X iff for each $x \in X \setminus C$ there are $y \in C$ and finitely many $x_1, \ldots, x_n \in X$ such that $xUx_1U \ldots Ux_nUy$. E.g. if U^{-1} is well-founded, then the set of U-maximal elements is cofinal.

Proposition. *If U is transitive, then (X, U) is non-paradoxical iff the set of U-maximal elements is cofinal in X.*

This generalizes both the Liar and Yablo Paradoxes, the models of which are transitive without maximal elements.

Proposition. *If* U^{-1} *is well-founded, then* (X, U) *is non-paradoxical. Moreover, any* (X, T, U) *satisfying* Ξ *is uniquely determined by values of* T *on* U*-maximal elements.*

Consequently, if U^{-1} is well-founded and there is a unique U-maximal element x (e.g. if U^{-1} is extensional) then there are exactly two models (X, T, U) satisfying Ξ, depending on Tx or $\neg Tx$.

The following concept of hereditarily winning relations generalizes well-foundedness.

A game of two players (Forster, Saveliev):

Let (X, x_0, U) be a pointed model. The first player moves by choosing any $x_1 \in X$ with Ux_1x_0;
the second player (knowing about x_1) moves by choosing any $x_2 \in X$ with Ux_2x_1;
and so on.

A player *wins* iff he could choose an $x_n \in X$ such that the play is stopped. (Possibly, nobody wins: e.g. look at a cycle.)

(X, x_0, U) is *winning* for one of the players iff it admits a winning strategy for him.
U is *hereditarily winning* iff $(X \restriction y, y, U \restriction y)$ are winning for all $y \in X$.

Fact: Well-founded relations are hereditarily winning. There exist lots of non-well-founded hereditarily winning relations.

Proposition. *If U^{-1} is hereditarily winning, then (X, U) is non-paradoxical.*

Main Theorem. *The paradoxicality (and hence non-paradoxicality) is a Δ^1_1 but not elementary property.*

Sketch of proof.

1. (X, U) is non-paradoxical iff it satisfies the Σ^1_1 formula

$$\exists\, T\, \Big(T \text{ is a subset of } X \,\wedge\, \Xi^{X,T,U}\Big).$$

2. (X, U) is non-paradoxical iff it satisfies the Π^1_1 formula

$$\forall\, \pi\, \Big(\pi \text{ is a map of } X \text{ into itself } \;\wedge\; |\mathrm{ran}\,\pi| \leq 2 \\ \to\; (\exists\, T' \subseteq \mathrm{ran}\,\pi)\, (\exists\, U' \subseteq \mathrm{ran}\,\pi \times \mathrm{ran}\,\pi) \\ \Xi^{\mathrm{ran}\,\pi, T', U'} \;\wedge\; \pi \text{ is a bounded morphism} \\ \text{of } (X, U) \text{ onto } (\mathrm{ran}\,\pi, U')\Big).$$

Here π collapses (X, U), so (X, U) can be expanded to a model of Ξ by letting $T = \pi^{-1}T'$.

Hence, Δ^1_1.

Why not elementary (= not first-order axiomatizable)?

3. Let $x_{m,0} = x_{n,0}$ and $x_{m,i} \neq x_{n,j}$ whenever $(i,j) \neq (m,n)$, and define:

$$X_n = \{x_{n,i}\}_{i \in \mathbb{Z}_{2n+1}}, \;\; U_n = \{(x_{n,i}, x_{n,i+1})\}_{i \in \mathbb{Z}_{2n+1}},$$

$$X_\omega = \{x_{\omega,i}\}_{i \in \mathbb{Z}}, \;\; U_\omega = \{(x_{\omega,i}, x_{\omega,i+1})\}_{i \in \mathbb{Z}},$$

$$X = \bigcup_{n<\omega} X_n, \;\; U = \bigcup_{n<\omega} U_n,$$

$$X' = \bigcup_{n\leq\omega} X_n, \;\; U' = \bigcup_{n\leq\omega} U_n.$$

Then:

(i) (X,U) is paradoxical while (X',U') is not,
(ii) $(X,U) \prec (X',U')$ (an elementary submodel).

Hence, not elementary.

Classification of non-paradoxical models

The class of non-paradoxical models is covered by five classes consisting of models that can be expanded to models of Ξ of classes I (a, b, or c, no matter), II (a or b), III, IV, and V. However the covering classes are *not disjoint*:

Example.

$$* \longrightarrow *$$

$$* \longrightarrow *$$

expansion:

$$\circ \longrightarrow \bullet \qquad\qquad \circ \longrightarrow \bullet$$

$$\circ \longrightarrow \bullet \qquad\qquad \bullet \longrightarrow \circ$$

collapse:

$$\circ \longrightarrow \bullet \qquad\qquad \circ \longleftrightarrow \bullet$$

$$\text{II} \qquad\qquad\qquad \text{III}$$

For $X_1, \ldots, X_n \in \{\mathrm{I, II, III, IV, V}\}$, let

$$X_1 + \ldots + X_n$$

denote the class of non-paradoxical models such that for each $i \in \{1, \ldots, n\}$ the model can be expanded to a model of Ξ collapsed to X_i.

E.g. the model above is in $\mathrm{II} + \mathrm{III}$.

Theorem. *There exist exactly eleven disjoint classes of form $X_1 + \ldots + X_n$, listed below:*

$$\mathrm{I}, \quad \mathrm{II}, \quad \mathrm{III}, \quad \mathrm{IV}, \quad \mathrm{V},$$

$$\mathrm{II} + \mathrm{III}, \quad \mathrm{II} + \mathrm{IV}, \quad \mathrm{III} + \mathrm{V}, \quad \mathrm{IV} + \mathrm{V},$$

$$\mathrm{III} + \mathrm{IV} + \mathrm{V}, \quad \mathrm{II} + \mathrm{III} + \mathrm{IV} + \mathrm{V}.$$