The reverse mathematics of Ekeland's variational principle

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The original Ekeland's variational principle

Theorem (Ekeland; J. Mathematical Analysis and Applications, 1974)

Let

- $\mathcal{X}$ *be a complete metric space;*
- $V\colon \mathcal{X} \to \mathbb{R}^{\geq 0}$ *be a continuous function;*
- $\varepsilon > 0$ *and* $y \in \mathcal{X}$ *be such that*
$$\inf(V) \leq V(y) \leq \inf(V) + \varepsilon;$$
- $\lambda > 0$.

Then there is an $x_* \in \mathcal{X}$ *such that*

- $V(x_*) \leq V(y)$*;*
- $d(x_*, y) \leq \lambda$*;*
- *for all* $w \neq x_*$*,*
$$V(x_*) < V(w) + \frac{\varepsilon}{\lambda} d(x_*, w).$$

Digestible Ekeland's variational principle (plus terminology)

Terminology:

- Henceforth, a **metric space** is a complete, separable metric space.
- For a metric space $\mathcal{X}$, we call a continuous $V\colon \mathcal{X} \to \mathbb{R}^{\geq 0}$ a **potential**.

Definition

Let $\mathcal{X}$ be a metric space, and let $V\colon \mathcal{X} \to \mathbb{R}^{\geq 0}$ be a potential. A **critical point** of V is a point $x_* \in \mathcal{X}$ such that, for all $y \neq x_*$,

$$V(x_*) < V(y) + d(x_*, y)$$

Theorem (Critical point theorem / digestible Ekeland's principle)

If $\mathcal{X}$ is a metric space and $V\colon \mathcal{X} \to \mathbb{R}^{\geq 0}$ is a potential, then V has a critical point.

What does Ekeland's variational principle do?

From Ekeland's own abstract, there are applications to:

- **Plateau's problem** (of finding a minimal surface with a given boundary),
- **partial differential equations**,
- **nonlinear eigenvalues**,
- **geodesics on infinite-dimensional manifolds**, and
- **control theory**.

Basically, Ekeland's variational principle is used to find approximate solutions to various optimization problems.

We care about using Ekeland's variational principle to find **fixed points**!

In particular, Ekeland's variational principle implies **Caristi's fixed point theorem**.

Caristi's fixed point theorem

Definition

A **Caristi system** is a triple $(\mathcal{X}, V, f)$, where

- $\mathcal{X}$ is a metric space,
- $V\colon \mathcal{X} \to \mathbb{R}^{\geq 0}$ is a potential, and
- $f\colon \mathcal{X} \to \mathcal{X}$ is an **arbitrary function**

such that

$$(\forall x \in \mathcal{X})[d(x, f(x)) \leq V(x) - V(f(x))].$$

Theorem (Caristi; TAMS 1976)

If $(\mathcal{X}, V, f)$ is a Caristi system, then f has a fixed point.

A critical point x_* of V is a fixed point of f:
If $f(x_*) \neq x_*$, then $V(x_*) - V(f(x_*)) < d(x_*, f(x_*))$, contradiction!

Lower semi-continuous functions

In Ekeland's variational principle, the potential $V\colon \mathcal{X} \to \mathbb{R}^{\geq 0}$ is in fact allowed to be **lower semi-continuous**.

Definition

Let $\mathcal{X}$ be a metric space. $V\colon \mathcal{X} \to \mathbb{R}$ is **lower semi-continuous** if

$$(\forall x \in \mathcal{X})(\forall \epsilon > 0)(\exists \delta > 0)(\forall y \in \mathcal{X})[d(x,y) < \delta \to f(y) \geq f(x) - \epsilon].$$

Lower semi-continuous functions can be complicated. Let $\mathcal{X}$ be a metric space, and let $\mathcal{C} \subseteq \mathcal{X}$ be closed. Then

$$V(x) = \begin{cases} 0 & \text{if } x \in \mathcal{C} \\ 1 & \text{if } x \notin \mathcal{C} \end{cases}$$

is lower semi-continuous.

Ekeland's principle, Caristi's theorem, and reverse math

We want to analyze the strengths of Ekeland's variational principle, Caristi's theorem, and related statements in the style of reverse mathematics.

But why?

- These theorems seem important, and their proofs are interesting.
- It's nice to study theorems that are not quite so old as the theorems that people (or at least I) usually study in reverse mathematics.
- The strengths of these theorems vary a lot depending on exactly what statement you care about.

A one-slide-introduction to reverse mathematics

Q: How strong is my theorem?

A: What do you mean?

. . . *thinking* . . . *thinking* . . .

Q: How strong is my sentence in the language of second-order arithmetic relative to a pre-specified base theory?

A: We can work with that.

The **typical situation** in reverse mathematics is:

- Consider two sentences φ and ψ in the language of second-order arithmetic (often expressing two well-known theorems).
- Does $\mathsf{RCA}_0 \vdash \varphi \to \psi$?

The Big Five subsystems of second-order arithmetic

We have two sorts: natural numbers and sets of natural numbers.
Ignore induction and focus on set-existence axioms.

RCA_0 says that sets computable from existing sets exist. Formally, Δ^0_1 comprehension.

WKL_0 adds the statement "every infinite subtree of $2^{<\mathbb{N}}$ has an infinite path" to RCA_0.

ACA_0 says that every arithmetical formula defines a set. Formally, arithmetical comprehension. (Intuition: ACA_0 can earn an **undergraduate degree in mathematics**.)

ATR_0 says that arithmetical comprehension can be iterated along a well-order.

Π^1_1-CA_0 says that every Π^1_1 formula defines a set. Formally, Π^1_1 comprehension.

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I said that we can only talk about natural numbers and sets of natural numbers.

But I want to talk about:

- trees
- real numbers
- metric spaces
- open and closed subsets of metric spaces
- continuous and lower semi-continuous functions
- and more!

This takes a lot of coding. I'll only say a few things about it.

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A **real number** is a rapidly converging Cauchy sequence of rationals.

A **point in a metric space** is a rapidly converging Cauchy sequence of points in a pre-specified countable dense set.

An **open set** is an enumeration of rational open balls.

A **continuous function** $f\colon \mathcal{X} \to \mathcal{Y}$ is an enumeration of pairs of rational open balls $\langle B_p(a), B_q(b)\rangle$ indicating that $f(B_p(a)) \subseteq \overline{B_q(b)}$.

A **lower semi-continuous function** $f\colon \mathcal{X} \to \mathbb{R}$ is an enumeration of pairs $\langle B_p(a), q\rangle$ indicating that $f(B_p(a)) \subseteq [q, \infty)$.

Remember what we're talking about?

Remember that we have

- $\mathcal{X}$, a metric space, and
- $V\colon \mathcal{X} \to \mathbb{R}^{\geq 0}$, a potential (i.e., a (lower semi-)continuous function).

A **critical point** of V is a point $x_* \in \mathcal{X}$ such that, for all $y \neq x_*$,

$$V(x_*) < V(y) + d(x_*, y).$$

Ekeland's principle: V has a critical point.

Equivalently, "x_* is a critical point of V" as means that, for all y,

$$[d(x_*, y) \leq V(x_*) - V(y)] \to y = x_*.$$

Sketching a proof of Ekeland's principle (continuous V)

Theorem (Critical point theorem / Ekeland's principle)

If $\mathcal{X}$ is a metric space and $V : \mathcal{X} \to \mathbb{R}^{\geq 0}$ is a potential, then V has a critical point (i.e., an x_* s.t. $\forall y[d(x_*, y) \leq V(x_*) - V(y) \Rightarrow y = x_*]$).

Build a sequence $(x_n : n < \omega)$ of points in $\mathcal{X}$:

- Choose any $x_0 \in \mathcal{X}$.
- Let $S_{x_n} = \{y \in \mathcal{X} : d(x_n, y) \leq V(x_n) - V(y)\}$.
- Choose $x_{n+1} \in S_{x_n}$ so that $V(x_{n+1}) \leq \Big(\inf_{y \in S_{x_n}} V(y)\Big) + 2^{-n}$.
- Notice $V(x_0) \geq V(x_1) \geq V(x_2) \geq \dots$, so let $c = \lim_{n\to\infty} V(x_n)$.
- Show that $d(x_m, x_n) \leq \sum_{i=m}^{n-1} d(x_i, x_{i+1}) \leq V(x_m) - c$.
- This means that $(x_n : n < \omega)$ is Cauchy. Let x_* be the limit.
- Show that x_* is a critical point.

Reverse math of Ekeland's principle (continuous V)

Theorem (F-D S T Y)

The following are equivalent over RCA_0*.*

(i) ACA_0.

(ii) *Ekeland's principle for* **arbitrary metric spaces** $\mathcal{X}$ *and* **continuous potentials** V.

A proof similar to the previous sketch is possible in ACA_0.

Theorem (F-D S T Y)

The following are equivalent over RCA_0*.*

(i) WKL_0.

(ii) *Ekeland's principle for* **compact metric spaces** $\mathcal{X}$ *and* **continuous potentials** V.

If $\mathcal{X}$ is compact, then extreme value theorem $\Rightarrow$ Ekeland's principle.

In both theorems, the reversals follow from reversals of Caristi's theorem.

Dealing with lower semi-continuous potentials

Let $\mathcal{X}$ be a metric space. Let $V \colon \mathcal{X} \to \mathbb{R}^{\geq 0}$ be lower semi-continuous.

Idea: Replace V with its **2-envelope**:

$$V_2(x) = \inf_{y \in \mathcal{X}} (V(y) + 2d(x, y))$$

Then

- V_2 is continuous;
- if x_* is a critical point of V_2, then $V_2(x_*) = V(x_*)$;
- if x_* is a critical point of V_2, then x_* is a critical point of V.

However, to define V_2, we need the set

$$\{\langle a, r, q \rangle : (\forall x \in B_r(a))(V(x) \geq q)\}.$$

- If $\mathcal{X}$ is compact, ACA_0 suffices.
- If $\mathcal{X}$ is not compact, then we need Π^1_1-CA_0.

Ekeland's principle in compact spaces with l.s.c. potentials

Theorem (F-D S T Y)

The following are equivalent over RCA_0*.*

(i) ACA_0.

(ii) *Ekeland's principle for* **compact metric spaces** $\mathcal{X}$ *and* **l.s.c. potentials** V.

For the reversal, use that the monotone convergence theorem is equivalent to ACA_0 over RCA_0.

- Let $q_0 < q_1 < q_2 < \cdots$ be an increasing sequence of rationals in $[0, 1]$.
- Define $V\colon [0, 1] \to \mathbb{R}^{\geq 0}$ by

$$V(x) = \begin{cases} 2 & \text{if } x < q_n \text{ for some } n \\ x & \text{otherwise.} \end{cases}$$

- Check that if x_* is a critical point of V, then $x_* = \sup\{q_n : n \in \mathbb{N}\}$.

In general, Ekeland's principle is equivalent to Π^1_1-CA_0

(Because lower semi-continuous functions are bad.)

Theorem (F-D S T Y)

The following are equivalent over RCA_0.

(i) Π^1_1-CA_0.

(ii) *Ekeland's principle for* **arbitrary metric spaces** $\mathcal{X}$ *and* **l.s.c. potentials** V.

The reversal takes advantage of the following fact.

Fact (see Simpson's SoSOA)

The following are equivalent over RCA_0.

(i) Π^1_1-CA_0.

(ii) *For every sequence* $(T_i : i \in \mathbb{N})$ *of subtrees of* $\mathbb{N}^{<\mathbb{N}}$*, there is a set* X *such that* $\forall i(i \in X \leftrightarrow T_i$ *is ill-founded*$)$.

Ekeland's principle is equivalent to Π^1_1-CA_0

Fact (see Simpson's SoSOA)

Π^1_1-CA_0 *is equivalent to the statement "for every sequence* $(T_i : i \in \mathbb{N})$ *of subtrees of* $\mathbb{N}^{<\mathbb{N}}$*, there is a set* X *such that* $\forall i(i \in X \leftrightarrow T_i$ *is ill-founded*)."

Work over ACA_0 (because the critical point theorem implies ACA_0).

- Let $(T_i : i \in \mathbb{N})$ be a sequence of subtrees of $\mathbb{N}^{<\mathbb{N}}$.
- Let $\mathcal{X} = \mathbb{N}^{\mathbb{N}}$.
- (For an $f \in \mathbb{N}^{\mathbb{N}}$, let $(f)_i$ denote $(f)_i(n) = f(\langle i, n \rangle)$.)
- Let $V : \mathbb{N}^{\mathbb{N}} \to \mathbb{R}^{\geq 0}$ be
$$V(f) = \sum \{2^{-i} : (f)_i \notin [T_i]\}.$$
- V is lower semi-continuous, and ACA_0 can make sense of this.
- Let f_* be a critical point, and let $X = \{i : (f_*)_i \in [T_i]\}$.
- Show that $i \in X$ if and only if T_i is ill-founded.

Thank you!

Thank you for coming to my talk!
Do you have a question about it?