Simplicial semantics of modal predicate logics

Valentin Shehtman

Wormshop 2017
October 18, Moscow

PLAN

- Introduction
- Simplicial sets
- Simplicial frames and models
- Soundness theorem
- Maximality theorem
- Kripke bundle semantics and Ghilardi's functor semantics
- Metaframe semantics
- Canonical simplical model and completeness theorems
- Incompleteness theorem

Introduction

Unlike the propositional case, in first-order modal (and intuitionistic) logic there is a gap between syntax and semantics. The standard Kripke semantics does not work properly in the predicate case - "most of" modal predicate logics are Kripke-incomplete.

Incompleteness of Kripke semantics was discovered quite long ago (H. Ono, 1973). After that a sequence of generalizations were proposed:

Kripke frames << *Kripke sheaves* << *Kripke bundles* << *Ghilardi's frames* << *Metaframes* << *Simplicial frames*

Moving along this sequence we regain completeness, but lose geometric intuition.

In this talk we give a brief overview of results in simplicial semantics and add some geometric motivation.

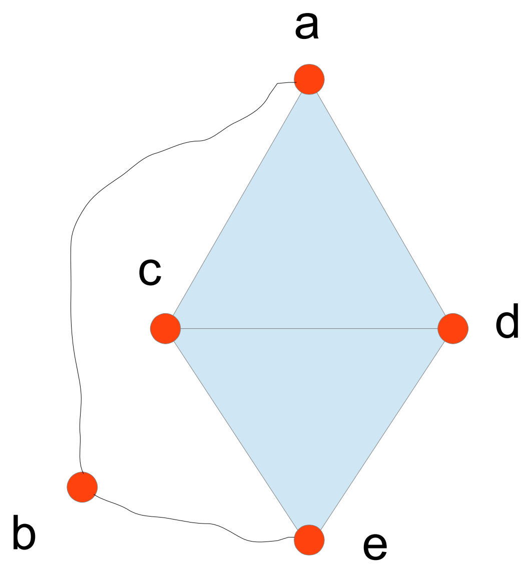
References

D.Gabbay, V. Shehtman, D. Skvortsov. Quantification in Nonclassical Logic, Volume 1. Elsevier, 2009.

D. Skvortsov, V. Shehtman. Maximal Kripke-type semantics for modal and superintuitionistic predicate logics. Annals of Pure and Applied Logic, 63:69-101, 1993.

Simplicial sets

Geometric simplicial complex



Abstract simplicial complex

{acd, cde, ac, ad, cd, de, ce, ab, be, a,b,c,d,e}

$$X \in S \ \& \ Y \subset X \Rightarrow Y \in S$$

Simplicial sets

$\mathbf{\Delta}$ is the category:

Ob $\mathbf{\Delta}$ = ω,

$\mathbf{\Delta}(m,n)$ = (non-strict) monotonic maps $m \to n$

A *simplicial set* is a contravariant functor X: $\mathbf{\Delta}^{\circ} \rightsquigarrow$ **SET**

X(n) is the set of n-dimensional simplices

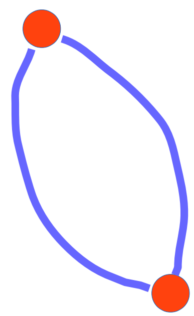
For every $f \in \mathbf{\Delta}(m,n)$, $X(f): X(n) \to X(m)$ is a face map selecting an m-dimensional face of an n-dimensional simplex (it may be degenerate – if f is not injective)

Example: If $a \in \mathbf{X}(2)$ is a triangle,

$f \in \mathbf{\Delta}(1,2)$, $f(0)=0$, $f(1)=2$, then X(f) chooses the second side of a (it can be denoted by a_{02}) .

Two differences between simplicial complexes and simplicial sets:

- simplicial sets include degenarate simplices
- in simplicial sets two different simplices may have the same proper faces:



Formulas

Modal predicate formulas are built from the following ingredients:

- the countable set of individual variables $Var=\{v_1,v_2,\ldots\}$
- countable sets of n-ary predicate letters (for every $n \geq 0$)
- $\rightarrow, \bot, \vee, \wedge, \Box$.
- $\exists, \forall$

The connectives $\neg, \Diamond$ are derived.

No constants or function symbols

NOTATION for the set of formulas: MF

Variable and formula substitutions

$[y_1,\ldots, y_n /x_1,\ldots, x_n]$ simultaneously replaces all free occurrences of $x_1,\ldots, x_n$ with $y_1,\ldots, y_n$ (renaming bound variables if necessary)

To obtain $[C(x_1,\ldots, x_n,y_1,\ldots, y_m)/P(x_1,\ldots, x_n)]A$ from A

(1) rename all bound variables of A that coincide with the "new" parameters $y_1,\ldots, y_m$ of C,

(2) replace every occurrence of every atom $P(z_1,\ldots, z_n)$ with $[z_1,\ldots, z_n /x_1,\ldots, x_n]C$

Strictly speaking, all substitutions are defined up to

congruence: formulas are congruent if they can be obtained by "legal" renaming of bound variables

$[Q(x,y,z)/P(x)]\ (\exists yP(y) \wedge P(z)) = \exists xQ(x,y,z) \wedge Q(z,y,z)$ or

$\exists uQ(u,y,z) \wedge Q(z,y,z)$

Modal logics

An modal predicate logic (mpl) is a set L of modal formulas such that L contains

- the classical propositional tautologies
- the axiom of **K:** $\Box(p \to q) \to (\Box p \to \Box q)$
- the standard predicate axioms

L is closed under the rules

- Modus Ponens: $A, A \to B \,/\, B$
- Necessitation: $A \,/\, \Box A$
- Generalization: $A \,/\, \forall x A$
- Substitution: A/SA (for any formula substitution S)

Modal propositional logics can be regarded as fragments of predicate logics (with only 0-ary predicate letters, without quantifiers).

Some notation

L+Γ := the smallest logic containing (L and Γ)

K := the minimal modal propositional logic

QL := the minimal predicate logic containing the propositional logic L

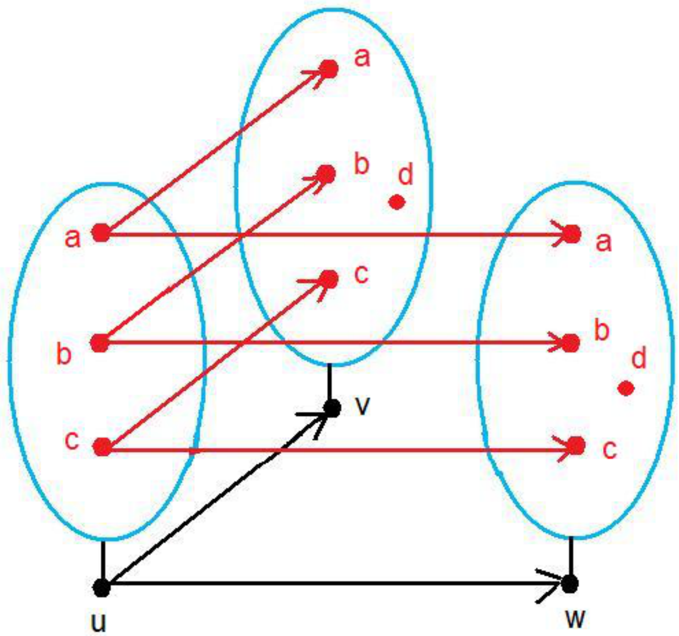
Kripke frame semantics for predicate logics

- A propositional Kripke frame $F=(W, R)$ ($W \neq \varnothing$, $R \subseteq W^2$)
- A predicate Kripke frame: $\Phi=(F,D)$, where

$D=(D_u)_{u \in W}$ is an expanding family of non-empty sets:

$$\text{if } u \, R \, v, \text{ then } D_u \subseteq D_v$$

D_u is the domain at the world u (consists of existing individuals).



A Kripke model over Φ is a collection of classical models:

M=(Φ,θ), where $\theta=(\theta_u)_{u\in W}$ is a valuation

$\theta_u(P)$ is an n-ary relation on D_u for each n-ary predicate letter P

For every modal formula $A(x_1,\ldots, x_n)$ and $d_1,\ldots, d_n \in D_u$

consider a D_u-sentence $A(d_1,\ldots, d_n)$.

<u>Def</u> Forcing (truth) relation $M,u \vDash B$

between the worlds u and D_u-sentences B is defined by induction:

- $M,u \vDash P(d_1,\ldots, d_n)$ iff $(d_1,\ldots, d_n) \in \theta_u(P)$
- $M,u \vDash a=b$ iff a equals b
- $M,u \vDash \Box B$ iff for any v, uRv implies $M,v \vDash B$
- $M,u \vDash \forall x\, B$ iff for any $d \in D_u$ $M,u \vDash [d/x]B$

etc. (the other cases are clear)

Def (truth in a Kripke model; validity in a frame)

$M \vDash A(x_1,\ldots,x_n)$ iff for any $u \in W$ $M,u \vDash \forall x_1 \ldots \forall x_n A(x_1,\ldots,x_n)$

$\Phi \vDash A$ iff for any M over Φ, $M \vDash A$

Soundness theorem

ML$(\Phi) := \{A \in MF \mid \Phi \vDash A\}$ is an mpl

Logics of this form are called *Kripke-complete*

Kripke sheaves

Kripke sheaves are an equivalent version of KFEs. They are obtained from KFEs by identifying equivalent individuals at every world.

Def A Kripke sheaf over a propositional reflexive transitive frame F=(W,R) is a triple $\mathscr{F}=(F,D,\rho)$, in which (F,D) is a predicate Kripke frame, $\rho=(\rho_{uv})_{uRv}$ is a collection of *transition maps* ("cross-reference") $\rho_{uv}: D_u \to D_v$ such that:

ρ_{uu} is the identity function on D_u

if uRvRw, then ρ_{uw} is the composition $\rho_{vw}\,\rho_{uv}$

Def. For an arbitrary propositional frame $F=(W,R)$, consider the transitive reflexive closure $F^*:=(W, R^*)$ (uR^*v iff there is an oriented path from u to v in F:

$uRw_1 \dots w_kRv$)

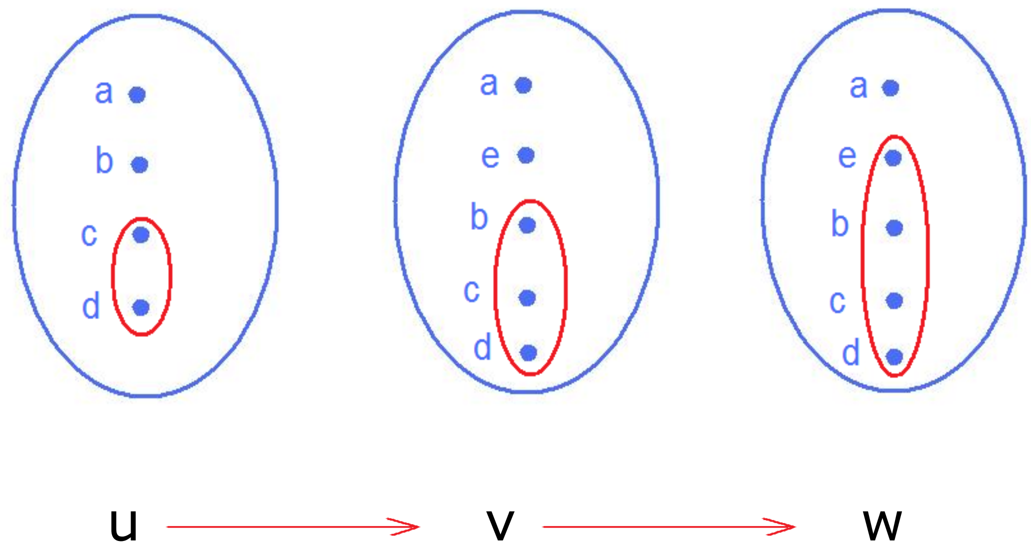
A Kripke sheaf over F is a triple $\mathscr{F}= (F,D,\rho)$, for which (F^*,D,ρ) is a Kripke sheaf over F^*.

Kripke sheaf models and forcing are defined in a straightforward way:

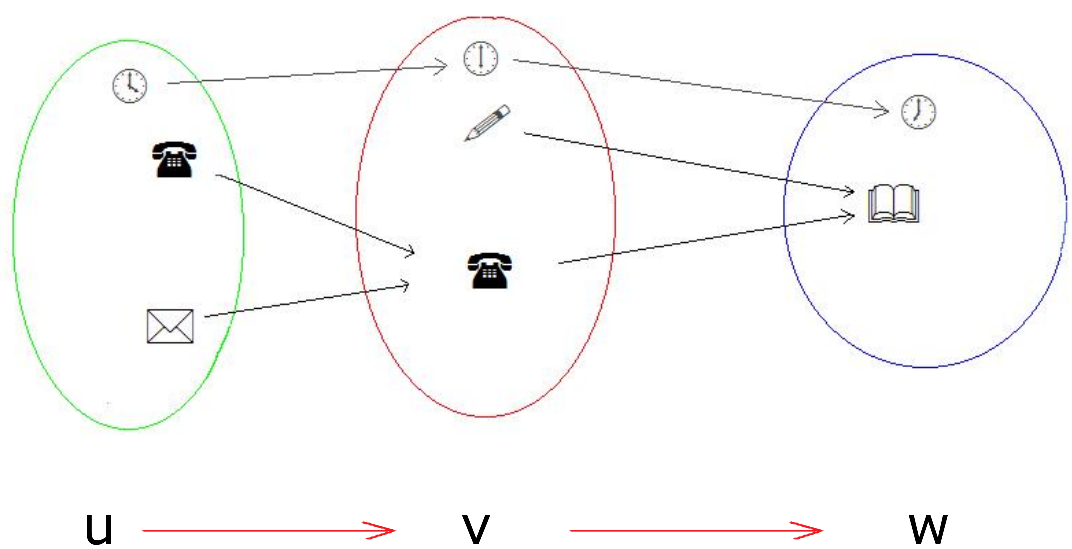
$M,u \models a=b$ iff a equals b

$M,u \models \Box B(d_1,\ldots, d_n)$ iff for any v, uRv implies

$$M,v \models B(\rho_{uv}(d_1),\ldots,\rho_{uv}(d_n))$$



a
b
c
d
a
e
b
c
d
a
e
b
c
d
u
v
w



u
v
w

Examples of Kripke-completeness

1. Surprisingly, for logics of the form **QL** not so many examples are known:

- for standard logics L (classical results by Kripke, Gabbay, Cresswell et al.)

  modal **K, T, D, B, K4, S4, S5**

  T: reflexive frames

  D: serial frames

  K4: transitive frames

  B: symmetric frames

  intuitionistic logic **H**

- for other cases, with more sophisticated proofs

  $\mathbf{S4.2} = \mathbf{S4} + \Diamond\Box A \rightarrow \Box\Diamond A$ (Ghilardi) confluent frames

$\mathbf{K4.3} = \mathbf{K4} + \Box(\Box A \wedge A \to B) \vee \Box(\Box B \wedge B \to A)$

non-branching transitive

$\mathbf{S4.3} = \mathbf{K4.3} + \Box A \to A$

$\mathbf{K4.3} + \Box\Box A \to \Box A$ density

(Corsi, 1990s)

2. For other kinds of logics see our book, Ch.6.

Barcan formula

$$Ba := \Diamond \exists x A \to \exists x \Diamond A$$

This formula is valid in a Kripke frame iff the domains *remain constant:*

if uRv then $D_u = D_v$

For he same basic cases, **QL**+Ba are also Kripke-complete

(but Ba is derivable in **QB, QS5**)

However, **QS4.2** + Ba is *K*-incomplete (SS 1990)

Def A propositional modal logic is called universal if the class of its frames is universal, i.e., the class of models of a universal classical first-order theory.

A propositional logic of a single finite frame is called tabular.

Theorem (Tanaka - Ono, 2001; book09) *If a modal propositional logic Λ is universal or tabular and K -complete, then $L = Q\Lambda + Ba$ is also K -complete.*

Simplicial frames

Introduced by Dmitry Skvortsov (1990); the first publication (abstract) in 1991; the paper in 1993.

In these publications simplicial frames we called 'Kripke metaframes'. Later the names were changed:

Kripke metaframes >> Simplicial frames

Cartesian metaframes >> Kripke metaframes

A simplicial frame is a modification of a simplicial set.

- $\boldsymbol{\Delta}$ is replaced by another category $\boldsymbol{\Sigma}$

  Ob $\boldsymbol{\Sigma} = \omega$,

  $\boldsymbol{\Sigma}_{mn}$ = *all* maps $I_m \to I_n$ (where $I_n = \{1,\ldots,n\}$, $I_0 = \varnothing$)

Let $\Sigma = \cup\{ \boldsymbol{\Sigma}_{mn} \mid m,n \geq 0\}$

- Accessibility relations are also involved

Roughly, a simplicial frame is a layered Kripke frame.

Definition

Let F=(W,R) be a propositional Kripke frame. A *simplicial frame* over F is **F**=(F, **D**, **R**, $\boldsymbol{\pi}$), where

- **D**=$(D^n)_{n\geq 0}$, **R**=$(R^n)_{n\geq 0}$, (D^n, R^n) is a propositional frame,

  (D^0, R^0) = F,

- $\boldsymbol{\pi} = (\pi_\sigma)_{\sigma\in\Sigma}$ is a family of maps

$$\pi_\sigma : D^n \rightarrow D^m \text{ for } \sigma \in \Sigma_{mn}$$

A *Kripke metaframe* is a simplicial frame, in which D^n is the n-th Cartesian power of D^1 and

$$\pi_\sigma(\mathbf{a}) = \mathbf{a} \cdot \sigma := (a_{\sigma(1)}, \ldots, a_{\sigma(m)}).$$

Informally:

D^0=W is the set of *possible worlds ((-1)-simplices)*

D^1 is the set of *individuals (0-simplices)*

D^n is the set of abstract n-tuples ((n-1)-simplices)

Definition A *valuation* in **F** is a function ξ such that

$\xi(P) \subseteq D^n$ for every n-ary predicate letter P.

M=(**F**, ξ) is a *simplicial model* over **F**.

An *assignment* of length n is a pair (**x, a**), where x is a list of different variables of length n, $\mathbf{a} \in D^n$.

Definition (truth of a formula in a simplicial model under an assignment involving the formula parameters)

$M,\mathbf{a} \vDash P(\mathbf{x}\cdot \sigma)\,[\mathbf{x}]$ iff $\pi_\sigma(\mathbf{a}) \in \xi(P)$

$M,\mathbf{a} \vDash \Box B\,[\mathbf{x}]$ iff for any $\mathbf{b}$, $\mathbf{a}R^n\mathbf{b}$ implies $M,\mathbf{b} \vDash B$

$M,\mathbf{a} \vDash \exists y\, B\,[\mathbf{x}]$ (with $y \notin \mathbf{x}$) iff for some $\mathbf{c} \in D^{n+1}$

$(\pi_\sigma(\mathbf{c}) = \mathbf{a}\ \&\ M,\mathbf{c} \vDash B\,[\mathbf{x}y]$),

where σ is the inclusion map $I_n \to I_{n+1}$

$M,\mathbf{a} \vDash \exists x_i\, B\,[\mathbf{x}]$ iff $M,\pi_\sigma(\mathbf{a}) \vDash \exists x_i\, B\,[\mathbf{x} \cdot \sigma]$,

where σ: $I_{n-1} \to I_n$ skips i.