On axiomatization and polytime decidability of strictly positive fragment of K4.3

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Definitions

Definition

- A formula is called *strictly positive* (SP-formula), if it is built from variables and the constant $\top$ only with $\wedge$ and $\Diamond$.
- A *sequent* is an implication $\alpha \to \beta$, where α and β are SP-formulas.
- The *SP-fragment* of a logic *L* is the set of all sequents provable in *L*.

Canonical trees

Definition

A *canonical tree* for any SP-formula α is a Kripke model, which is denoted $T[\alpha]$ and is defined by induction:

- If α is a variable, then $T[\alpha]$ is a single-point irreflexive model and α is the only variable true at this point.
- If $\alpha = \top$, then $T[\alpha]$ is a single-point irreflexive model and all the variables are false at this point.
- If $\alpha = \beta \wedge \gamma$, then $T[\alpha]$ is $T[\beta]$ and $T[\gamma]$ joined by uniting their roots. If a variable was true at any of the old roots, it is true at the new root.
- If $\alpha = \Diamond\beta$, then $T[\alpha]$ is $T[\beta]$ united with a new point, at which all variables are false and which is connected with every point in $T[\beta]$.

Previous Results

Lemma (Dashkov)

Let $\mathcal{M}$ be a transitive Kripke model, and α a SP-formula. Then $\mathcal{M}, x \models \alpha$ iff there's a homomorphism $f\colon T[\alpha] \to \mathcal{M}$ such that $f(r(\alpha)) = x$.

Theorem (Dashkov)

For all SP-formulas α, β, the following are equivalent:

(i) $K4 \vdash \alpha \to \beta$

(ii) $T[\alpha], r(\alpha) \models \beta$

(iii) *There's a homomorphism $f\colon T[\beta] \to T[\alpha]$, such that $f(r(\beta)) = r(\alpha)$.*

Linearizations

Definition

A Kripke model T is called a *linearisation* of $T[\alpha]$, if there exists a surjective homomorphism $f\colon T[\alpha] \to T$, and the relation in T is a linear irreflexive order.

Lemma

For any SP-formula α we have SP-formulas $\alpha_1, \dots, \alpha_n$, such that every $T[\alpha_i]$ is a linearisation of $T[\alpha]$ and $K4.3 \vdash \alpha \to \bigvee_{i=1}^{n} \alpha_i$.

Antichains completion

Definition

Let $(W, <)$ be a partial order. An *antichain* in W is a nonempty subset of W, consisting of pairwise incomparable points.

Definition

Let $\mathcal{M} = (W, <, v)$ be a Kripke model. Its *antichains completion* is the following Kripke model $\mathcal{M}^\Upsilon = (W^\Upsilon, \prec, v^\Upsilon)$:

- W^Υ is the set of all antichains in W.
- $x \prec y \iff \forall\, b \in y\, \exists\, a \in x\ : a < b$.
- $v^\Upsilon(p) = \{x \mid \forall\, a \in x\ a \in v(p)\}$.

Axiomatization of SP(K4.3)

Definition

$$comwit_n = \bigwedge_{i=1}^{n} \Diamond(p \wedge \Diamond q_i) \to \Diamond(p \wedge \bigwedge_{i=1}^{n} \Diamond q_i)$$

Theorem

For all SP-formulas α, β, the following are equivalent:

(i) $K4.3 \vdash \alpha \to \beta$

(ii) β *is true at the root of every linearisation of* $T[\alpha]$

(iii) $T[\alpha]^{\Upsilon}, \{r(\alpha)\} \models \beta$

(iv) $K4^{+} + comwit_2 \vdash \alpha \to \beta$.

New structure on $T[\alpha]^\Upsilon$

Definition

- $A \subset W$, $\Diamond A = \{b \in W \mid \exists a \in A : a > b\}$
- $(A_0, \{A_1, \ldots, A_k\}) \wedge (B_0, \{B_1, \ldots, B_m\}) := (A_0 \cap B_0, \{A_1 \cap B_0, A_2 \cap B_0, \ldots, A_k \cap B_0, B_1 \cap A_0, B_2 \cap A_0, \ldots, B_m \cap A_0\})$
- $\Diamond(A, \{A_1, A_2, \ldots, A_k\}) := (W, \{\Diamond A_1, \Diamond A_2, \ldots, \Diamond A_k\})$
- $g(p) = (v(p), \{v(p)\})$, $g(\top) = (W, \{W\})$,
  $g(\alpha \wedge \beta) = g(\alpha) \wedge g(\beta)$, $g(\Diamond\alpha) = \Diamond g(\alpha)$
- $f(A_0, \{A_1, \ldots, A_k\}) =$
  $\{X \in W^\Upsilon \mid X \subset A_0, \forall\, i = 1, 2, \ldots, k \;\; X \cap A_i \neq \emptyset\}$

Polytime decidability of SP(K4.3)

Theorem

Let X be an antichain in the model $\mathcal{M}^\Upsilon$, and α a SP-formula. Then $\mathcal{M}^\Upsilon, X \models \alpha$ iff $X \in f(g(\alpha))$.

Corollary

The SP-fragment of the logic K4.3 is polytime decidable.