The Second Incompleteness Theorem for Σ^0_1-semi-numerations

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Numerations and Semi-numerations

We consider theories U in finite signature with arbitrarily complex axiom sets X. Let $N : \mathsf{S}^1_2 \lhd U$. Consider any U-formula α.

We say that α *semi-numerates* X in N iff, for each $A \in X$, we have $U \vdash \alpha(\underline{\ulcorner A \urcorner})$. Here we use N-numerals.

We say that α *numerates* X in N iff, for each U-sentence A, we have $A \in X$ iff $U \vdash \alpha(\underline{\ulcorner A \urcorner})$.

Overview

Example

Let Z be any set of numbers. By a theorem independently due to Feferman, Scott and Kripke, there is a Σ^0_1-predicate $S^*(x)$ such that $\mathrm{EA} + \{S^*(\underline{n}) \mid n \in Z\} + \{\neg S^*(\underline{n}) \mid n \notin Z\}$ is consistent.

Consider the theory U axiomatized by $X := \mathrm{EA} + \{S^*(\underline{n}) \mid n \in Z\}$. Let β be a standard representation of the finitely many axioms of EA. We take $\zeta(x) := (\beta(x) \vee \exists y < x\, (x = \ulcorner S^*(\dot{y}) \urcorner \wedge S^*(y)))$.

Clearly,

- ζ numerates X in U,
- ζ is Σ^0_1,
- Z is 1-reducible to U.

So we have arbitrarily complex theories for which the axiom set is numerated by a Σ^0_1-formula (relative to the designated numbers).

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G2 for Σ^0_1 semi-numerations: stated

Theorem

Suppose:

- *U is consistent,*
- *U is axiomatized by X,*
- $N : \mathsf{EA} \lhd U$,
- *σ is Σ^0_1,*
- *σ^N semi-numerates X in N.*

Then, $U \nvdash (\Diamond_\sigma \top)^N$.

Alternative formulation of the conclusion: not $N : U \rhd (\mathsf{EA} + \Diamond_\sigma \top)$.

G2 for Σ^0_1 semi-numerations: proved

Proof.
Suppose $X \vdash (\Diamond_\sigma \top)^N$. Let X_0 be a finite subset of X such that $X_0 \vdash \mathrm{EA}^N$ and $X_0 \vdash (\Diamond_\sigma \top)^N$. Let $Y_0 := X_0 + \{\sigma^N(\ulcorner \underline{A} \urcorner) \mid A \in X_0\}$.

Using the fact that we have Σ^0_1-completeness in EA, we see that, for any $B \in Y_0$, we have $Y_0 \vdash \Box^N_\sigma B$.

Let η_0 be a standard representation of the finite set Y_0. We find: $Y_0 \vdash (\forall A \in \mathrm{sent}_U\ (\Box^N_{\eta_0} A \to \Box_\sigma A))^N$. Note that we do not need internal Σ^0_1-collection since Y_0 is standardly finite.

It follows that $Y_0 \vdash \Diamond^N_{\eta_0} \top$. But this contradicts the ordinary G2. ∎

There is an alternative proof using Craig's trick. It seems that I need that alternative version to prove the corresponding variant of Feferman's Theorem.

Do we have the theorem also for S^1_2?

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Example: what the example does

The formalized 4 axiom may fail in our setting. Thus the ordinary proof of G2 does not always work without an extra trick.

There is a Σ^0_1-formula $\sigma(x)$ with the following properties.

- σ defines the axioms of EA in the standard model, and, hence, numerates the axioms of EA in EA.
- EA knows that σ defines a finite set of axioms.
- EA knows that the theory defined by σ is between EA and EA $+ \Diamond_\beta \top$, where β is a standard axiomatization of EA.
- EA does not prove formalized G2 for σ.

Example: a quick peek at what it looks like 1

We work in EA. We define a Kripke model $\mathcal{K}$. Let p be any (possibly non-standard) number. Our model has nodes $0, \ldots, p+1$. We set $x \prec y$ iff $x = 0$ and $1 \leq y \leq p+1$.

Let C be a single axiom for EA and let $\beta(x) := x = \underline{\ulcorner C \urcorner}$.

We define the Solovay function h_p on $\mathcal{K}$ for β.

- $\ell_p = 0$ iff $\forall x\, h(x) = 0$.
- $\ell_p = y$, if $0 \prec y$ and $\exists x\, h(x) = y$.
- $h_p(0) = 0$,
- $h_p(y+1) := \begin{cases} x & \text{if } h(y) \prec x \text{ and } \mathrm{proof}_\beta(y, \ulcorner \ell_{\dot{p}} \neq \dot{x} \urcorner) \\ h_p(y) & \text{otherwise} \end{cases}$

Example: a quick peek at what it looks like 2

- def(s, y) is the usual Σ^0_1-predicate for: s is a Σ^0_1-formula that defines x.
- $\mathfrak{p}$ is a partial term for the smallest inconsistency-proof of PA.
- $\sigma(x) :\leftrightarrow \beta(x) \vee (\mathfrak{p}\downarrow \wedge \exists y \leq \mathfrak{p}\, (x = \ulcorner \ell_{\mathfrak{p}} \neq (y \dot{+} 1)\urcorner \wedge \exists s < \mathfrak{p}\, \mathrm{def}(s, y)))$.
- $\mathsf{S}^\star :\leftrightarrow (\mathfrak{p}\downarrow \wedge \forall y \leq \mathfrak{p}\, \exists s < \mathfrak{p}\, \mathrm{def}(s, y))$.

Note that $\mathsf{EA} + \mathfrak{p}\uparrow \vdash \forall x\, (\sigma(x) \leftrightarrow \beta(x))$ and $\mathsf{EA} \vdash \sigma(\underline{\ulcorner B \urcorner})$ iff $\mathsf{EA} \vdash \beta(\underline{\ulcorner B \urcorner})$.

Example: a quick peek at what it looks like 3

Selected Facts:

- $\mathsf{EA} \vdash \mathfrak{p} \to \Box_\beta \neg \mathsf{S}^\star$.
- $\mathsf{EA} \vdash (y \leq x \wedge \Box_\beta(\ell_x \neq y+1)) \to \Box_\beta \bot$.
- $\mathsf{EA} \vdash \bigvee_{y \leq x}(\ell_x = y+1) \leftrightarrow \Box_\beta \bot$.

Theorem

- $\mathsf{EA} \vdash \Box_\sigma \bot \leftrightarrow ((\mathsf{S}^\star \wedge \Box_\beta \Box_\beta \bot) \vee \Box_\beta \bot)$.
- $\mathsf{EA} \vdash \Box_\sigma \Diamond_\sigma \top \leftrightarrow (\mathsf{S}^\star \vee \Box_\beta \bot)$.

Using methods due to Paris & Kirby, we can build a model $\mathcal{N}$ of $\mathsf{EA} + \mathsf{S}^\star + \Diamond_\beta \Diamond_\beta \top$ and we are done.

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Thank You