Natural axiomatic theories are well-ordered by consistency strength.

*Ordinal analysis*: assign recursive ordinals to theories as a measurement of their consistency strength.

*Beklemishev’s method*: iterate consistency statements over a base theory until you reach the $\Pi^0_1$ consequences of the target theory.

Why are natural theories amenable to such analysis?
Natural Turing degrees are well-ordered by Turing reducibility.

\(0, 0′, \ldots, 0^\omega, \ldots, \emptyset, \ldots, 0^\# \ldots\)

**Martin’s Conjecture:** (AD) The non-constant degree invariant functions are pre-well-ordered by the relation

\[ \text{“} f(a) \leq_T g(a) \text{ for all } a \text{ in a cone of Turing degrees.} \text{”} \]

Moreover, the successor for this pre-well-ordering is induced by the Turing jump.
Our base theory is *elementary arithmetic*, $EA$, a subsystem of arithmetic just strong enough for usual arithmetization of syntax.

We focus on recursive functions $f$ that are *monotonic*, i.e.,

$$\text{if } EA \vdash \varphi \rightarrow \psi, \text{ then } EA \vdash f(\varphi) \rightarrow f(\psi).$$

Our goal is to show that $\varphi \mapsto (\varphi \land Con(\varphi))$ and its iterates are canonical monotonic functions.
Some notation

We write $\varphi \vdash \psi$ when $EA \vdash \varphi \rightarrow \psi$ and say that $\varphi$ implies $\psi$.

We say that $\varphi$ strictly implies $\psi$ if
(i) $\varphi \vdash \psi$ and
(ii) either $\psi \not\vdash \varphi$ or $\psi \vdash \bot$.

We write $[\varphi] = [\psi]$ if $\varphi \vdash \psi$ and $\psi \vdash \varphi$. 
Theorem (Montalbán–W.)

Let $f$ be monotonic. Suppose that for all $\varphi$, 
(i) $\varphi \land \text{Con}(\varphi)$ implies $f(\varphi)$, 
(ii) $f(\varphi)$ strictly implies $\varphi$.

Then for cofinally many true sentences $\varphi$,

$$EA \vdash f(\varphi) \iff (\varphi \land \text{Con}(\varphi)).$$

Corollary

There is no monotonic $f$ such that for every $\varphi$, 
(i) $\varphi \land \text{Con}(\varphi)$ strictly implies $f(\varphi)$ and 
(ii) $f(\varphi)$ strictly implies $\varphi$. 
Monotonicity is essential

Can we weaken the condition of *monotonicity*, i.e.,

\[ \text{if } EA \vdash \varphi \rightarrow \psi, \text{ then } EA \vdash f(\varphi) \rightarrow f(\psi), \]

to the condition of *extensionality*, i.e.,

\[ \text{if } EA \vdash \varphi \leftrightarrow \psi, \text{ then } EA \vdash f(\varphi) \leftrightarrow f(\psi)? \]

**Theorem (Shavrukov–Visser)**

*There is an extensional* \( f \) *such that for all* \( \varphi \),

(i) \( \varphi \land \text{Con}(\varphi) \) *strictly implies* \( f(\varphi) \) *and*

(ii) \( f(\varphi) \) *strictly implies* \( \varphi \).
Theorem (Visser)

For all $\varphi$, $EA \vdash Con_{CF}(Con_{CF}(\varphi)) \leftrightarrow Con(\varphi)$.

However, for all $\varphi$ that prove the cut-elimination theorem,

$$EA \vdash (\varphi \land Con(\varphi)) \leftrightarrow (\varphi \land Con_{CF}(\varphi)).$$

Similar considerations apply to the Friedman–Rathjen–Wiermann notion of *slow consistency*.

Question: Does the lattice of $\Pi^0_1$ sentences enjoy uniform monotonic density?
Given an elementary presentation of an ordinal $\alpha$, we define the iterates of $\text{Con}$ as follows.

\[
\text{Con}^0(\phi) := \top \\
\text{Con}^{\beta+1}(\phi) := \text{Con}(\phi \land \text{Con}^\beta(\phi)) \\
\text{Con}^\lambda(\phi) := \forall \beta < \lambda \text{Con}^\beta(\phi)
\]

N.B. $\text{Con}^1(\phi) = \text{Con}(\phi)$. 
Theorem (Montalbán–W.)

Let $f$ be monotonic. Suppose that for all $\varphi$,

(i) $\varphi \land \text{Con}^\alpha(\varphi)$ implies $f(\varphi)$,
(ii) $f(\varphi)$ strictly implies $\varphi \land \text{Con}^\beta(\varphi)$ for all $\beta < \alpha$.

Then for cofinally many true sentences $\varphi$,

$$EA \vdash f(\varphi) \iff (\varphi \land \text{Con}^\alpha(\varphi)).$$

Corollary

There is no monotonic $f$ such that for every $\varphi$,

(i) $\varphi \land \text{Con}^\alpha(\varphi)$ strictly implies $f(\varphi)$ and
(ii) $f(\varphi)$ strictly implies $\varphi \land \text{Con}^\beta(\varphi)$ for all $\beta < \alpha$. 
Iterates of $\text{Con}$ are inevitable.

**Theorem (Montalbán–W.)**

Let $f$ be a monotonic function such that for every $\varphi$, 
(i) $\varphi \land \text{Con}^n(\varphi)$ implies $f(\varphi)$ and 
(ii) $f(\varphi)$ implies $\varphi$.

Then for some $\varphi$ and some $k \leq n$, 

$$[f(\varphi)] = [\varphi \land \text{Con}^k(\varphi)] \neq [\bot].$$
The main theorem

Theorem (Montalbán–W.)

Suppose $f$ is monotonic and, for all $\varphi$, $f(\varphi) \in \Pi^0_1$. Then either
(i) for some $\varphi$, $(\varphi \land \text{Con}^\alpha(\varphi)) \not\models f(\varphi)$ or
(ii) for some $\beta \leq \alpha$ and $\varphi$, $[\varphi \land f(\varphi)] = [\varphi \land \text{Con}^\beta(\varphi)] \neq [\bot]$.

The proof of this theorem involves Schmerl’s technique of reflexive induction in a seemingly essential way.
The main theorem

Theorem (Montalbán–W.)

Suppose $f$ is monotonic and, for all $\varphi$, $f(\varphi) \in \Pi^0_1$. Then either
(i) for some $\varphi$, $(\varphi \land \text{Con}^\alpha(\varphi)) \nvdash f(\varphi)$ or
(ii) for some $\beta \leq \alpha$ and $\varphi$, $[\varphi \land f(\varphi)] = [\varphi \land \text{Con}^\beta(\varphi)] \neq [\bot]$.

The main theorem resembles the following theorem of Slaman and Steel.

Theorem (Slaman–Steel)

Suppose $f : 2^\omega \rightarrow 2^\omega$ is Borel, order-preserving with respect to $\leq_T$, and increasing on a cone. Then for any $\alpha < \omega_1$, either
(i) $(x^{(\alpha)} <_T f(x))$ on a cone or
(ii) for some $\beta \leq \alpha$, $f(x) \equiv_T x^{(\beta)}$ on a cone.
The main theorem

**Theorem (Montalbán–W.)**

Suppose $f$ is monotonic and, for all $\varphi$, $f(\varphi) \in \Pi_1^0$. Then either (i) for some $\varphi$, $(\varphi \land \text{Con}^\alpha(\varphi)) \not\models f(\varphi)$ or (ii) for some $\beta \leq \alpha$ and $\varphi$, $[\varphi \land f(\varphi)] = [\varphi \land \text{Con}^\beta(\varphi)] \neq [\bot]$.

Question: In case (ii), can we find a true $\varphi$ such that $[\varphi \land f(\varphi)] = [\varphi \land \text{Con}^\beta(\varphi)]$?
Recall: \( \varphi \) is \( 1\text{-consistent} \) if \( \text{EA} + \varphi \) is consistent with the true \( \Pi^0_1 \) theory of arithmetic.

\( 1\text{Con} \) is a \( \Pi^0_2 \) analogue of consistency.

Recall: \( 1\text{Con}(\top) \) is \( \Pi^0_1 \) conservative over \( \{ \text{Con}^k(\top) : k < \omega \} \).

Such conservativity results are drastically violated in the limit.

If \( \varphi \) implies \( \Pi^0_1 \) transfinite induction along \( \alpha \), then \( (\varphi \land 1\text{Con}(\varphi)) \) strictly implies \( (\varphi \land \text{Con}^\alpha(\varphi)) \).

Is \( 1\text{Con} \) the weakest such function?
The *Harrison linear order* $\mathcal{H}$ is a recursive linear order with no hyperarithmetic descending sequences.

$$\mathcal{H} \cong \omega_1^{CK} \times (1 + \mathbb{Q})$$

Thus, $\mathcal{H}$ provides a notation to each recursive ordinal.

Using Gödel’s fixed point lemma, we can iterate $Con$ along $\mathcal{H}$.
We say that $f$ majorizes $g$ if there is a true $\varphi$ such that whenever $\psi \vdash \varphi$ then $f(\psi)$ strictly implies $g(\psi)$.

**Theorem (Montalbán–W.)**

For every non-standard $\alpha \in \mathcal{H}$ and standard $\beta \in \mathcal{H}$,

(i) $\varphi \mapsto (\varphi \land \text{Con}^\alpha(\varphi))$ majorizes $\varphi \mapsto (\varphi \land \text{Con}^\beta(\varphi))$ but

(ii) $\varphi \mapsto (\varphi \land \text{1Con}(\varphi))$ majorizes $\varphi \mapsto (\varphi \land \text{Con}^\alpha(\varphi))$. 

James Walsh

On the inevitability of the consistency operator
We would like to strengthen our positive results by changing *cofinally* to *in the limit*.

Let \( f \) be recursive and monotonic. Suppose that for all \( \varphi \)

(i) \( \varphi \land \text{Con}(\varphi) \) implies \( f(\varphi) \) and
(ii) \( f(\varphi) \) implies \( \varphi \).

**Question:** Must \( f \) be equivalent to the identity or to \( \text{Con} \) on a true ideal?

**Question:** Is the relation of cofinal agreement on true sentences an equivalence relation on recursive monotonic operators?
Thanks!

Proof-theoretic analysis by iterated reflection.
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