On the inevitability of the consistency operator

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The consistency strength hierarchy

Natural axiomatic theories are well-ordered by consistency strength.

Ordinal analysis: assign recursive ordinals to theories as a measurement of their consistency strength.

Beklemishev's method: iterate consistency statements over a base theory until you reach the Π^0_1 consequences of the target theory.

Why are natural theories amenable to such analysis?

Natural Turing degrees are well-ordered by Turing reducibility.

$0, 0', ..., 0^{\omega}, ..., \mathcal{O}, ..., 0^{\sharp}, ...$

Martin's Conjecture: (AD) The non-constant degree invariant functions are pre-well-ordered by the relation

"$f(a) \leq_T g(a)$ *for all* a *in a cone of Turing degrees.*"

Moreover, the successor for this pre-well-ordering is induced by the Turing jump.

The setting

Our base theory is *elementary arithmetic*, *EA*, a subsystem of arithmetic just strong enough for usual arithmetization of syntax.

We focus on recursive functions f that are *monotonic*, i.e.,

$$\text{if } EA \vdash \varphi \rightarrow \psi, \textit{ then } EA \vdash f(\varphi) \rightarrow f(\psi).$$

Our goal is to show that $\varphi \mapsto (\varphi \wedge Con(\varphi))$ and its iterates are canonical monotonic functions.

Some notation

We write $\varphi \vdash \psi$ when $\mathsf{EA} \vdash \varphi \to \psi$ and say that φ *implies* ψ.

We say that φ *strictly implies* ψ if
(i) $\varphi \vdash \psi$ and
(ii) either $\psi \nvdash \varphi$ or $\psi \vdash \bot$.

We write $[\varphi] = [\psi]$ if $\varphi \vdash \psi$ and $\psi \vdash \varphi$.

Between the identity and *Con*

Theorem (Montalbán–W.)

Let f be monotonic. Suppose that for all φ,
(i) $\varphi \wedge Con(\varphi)$ implies $f(\varphi)$,
(ii) $f(\varphi)$ strictly implies φ.
Then for cofinally many true sentences φ,

$$EA \vdash f(\varphi) \leftrightarrow (\varphi \wedge Con(\varphi)).$$

Corollary

*There is **no** monotonic f such that for every φ,*
(i) $\varphi \wedge Con(\varphi)$ strictly implies $f(\varphi)$ and
(ii) $f(\varphi)$ strictly implies φ.

Monotonicity is essential

Can we weaken the condition of *monotonicity*, i.e.,

$$\textit{if } EA \vdash \varphi \rightarrow \psi, \textit{ then } EA \vdash f(\varphi) \rightarrow f(\psi),$$

to the condition of *extensionality*, i.e.,

$$\textit{if } EA \vdash \varphi \leftrightarrow \psi, \textit{ then } EA \vdash f(\varphi) \leftrightarrow f(\psi)?$$

Theorem (Shavrukov–Visser)

There is an extensional f such that for all φ,
(i) $\varphi \wedge Con(\varphi)$ strictly implies $f(\varphi)$ and
(ii) $f(\varphi)$ strictly implies φ.

Cut-free consistency

Theorem (Visser)

For all φ*,* $\mathsf{EA} \vdash Con_{CF}(Con_{CF}(\varphi)) \leftrightarrow Con(\varphi)$*.*

However, for all φ that prove the cut-elimination theorem,

$$\mathsf{EA} \vdash (\varphi \wedge Con(\varphi)) \leftrightarrow (\varphi \wedge Con_{CF}(\varphi)).$$

Similar considerations apply to the Friedman–Rathjen–Wiermann notion of *slow consistency*.

Question: Does the lattice of Π^0_1 sentences enjoy uniform monotonic density?

Iterates of *Con*

Given an elementary presentation of an ordinal α, we define the iterates of *Con* as follows.

$$Con^0(\varphi) := \top$$

$$Con^{\beta+1}(\varphi) := Con(\varphi \wedge Con^{\beta}(\varphi))$$

$$Con^{\lambda}(\varphi) := \forall \beta < \lambda Con^{\beta}(\varphi)$$

N.B. $Con^1(\varphi) = Con(\varphi)$.

Generalizations to the effective transfinite

Theorem (Montalbán–W.)

Let f be monotonic. Suppose that for all φ,
(i) $\varphi \wedge Con^{\alpha}(\varphi)$ implies $f(\varphi)$,
(ii) $f(\varphi)$ strictly implies $\varphi \wedge Con^{\beta}(\varphi)$ for all $\beta < \alpha$.
Then for cofinally many true sentences φ,

$$EA \vdash f(\varphi) \leftrightarrow (\varphi \wedge Con^{\alpha}(\varphi)).$$

Corollary

*There is **no** monotonic f such that for every φ,*
(i) $\varphi \wedge Con^{\alpha}(\varphi)$ strictly implies $f(\varphi)$ and
(ii) $f(\varphi)$ strictly implies $\varphi \wedge Con^{\beta}(\varphi)$ for all $\beta < \alpha$.

Iterates of *Con* are inevitable

Theorem (Montalbán–W.)

Let f be a monotonic function such that for every φ,
(i) $\varphi \wedge Con^n(\varphi)$ implies $f(\varphi)$ and
(ii) $f(\varphi)$ implies φ.
Then for some φ and some $k \leq n$,

$$[f(\varphi)] = [\varphi \wedge Con^k(\varphi)] \neq [\bot].$$

The main theorem

Theorem (Montalbán–W.)

Suppose f is monotonic and, for all φ, $f(\varphi) \in \Pi^0_1$. *Then either*
(i) for some φ, $(\varphi \wedge Con^{\alpha}(\varphi)) \nvdash f(\varphi)$ *or*
(ii) for some $\beta \leq \alpha$ *and* φ, $[\varphi \wedge f(\varphi)] = [\varphi \wedge Con^{\beta}(\varphi)] \neq [\perp]$.

The proof of this theorem involves Schmerl's technique of *reflexive induction* in a seemingly essential way.

The main theorem

Theorem (Montalbán–W.)

Suppose f is monotonic and, for all φ, $f(\varphi) \in \Pi^0_1$. Then either
(i) for some φ, $(\varphi \wedge Con^\alpha(\varphi)) \nvdash f(\varphi)$ or
(ii) for some $\beta \leq \alpha$ and φ, $[\varphi \wedge f(\varphi)] = [\varphi \wedge Con^\beta(\varphi)] \neq [\bot]$.

The main thorem resembles the following theorem of Slaman and Steel.

Theorem (Slaman–Steel)

Suppose $f : 2^\omega \to 2^\omega$ is Borel, order-preserving with respect to $\leq_T$, and increasing on a cone. Then for any $\alpha < \omega_1$, either
(i) $(x^{(\alpha)} <_T f(x))$ on a cone or
(ii) for some $\beta \leq \alpha$, $f(x) \equiv_T x^{(\beta)}$ on a cone.

The main theorem

Theorem (Montalbán–W.)

Suppose f *is monotonic and, for all* φ, $f(\varphi) \in \Pi^0_1$. *Then either*
(i) for some φ, $(\varphi \wedge Con^{\alpha}(\varphi)) \nvdash f(\varphi)$ *or*
(ii) for some $\beta \leq \alpha$ *and* φ, $[\varphi \wedge f(\varphi)] = [\varphi \wedge Con^{\beta}(\varphi)] \neq [\bot]$.

Question: In case (ii), can we find a *true* φ such that $[\varphi \wedge f(\varphi)] = [\varphi \wedge Con^{\beta}(\varphi)]$?

1-Consistency

Recall: φ is *1-consistent* if $\mathsf{EA} + \varphi$ is consistent with the true Π^0_1 theory of arithmetic.

$1Con$ is a Π^0_2 analogue of consistency.

Recall: $1Con(\top)$ is Π^0_1 conservative over $\{Con^k(\top) : k < \omega\}$.

Such conservativity results are drastically violated in the limit.

If φ implies Π^0_1 transfinite induction along α, then $(\varphi \wedge 1Con(\varphi))$ strictly implies $(\varphi \wedge Con^\alpha(\varphi))$.

Is $1Con$ the weakest such function?

Harrison Linear Order

The *Harrison linear order* $\mathcal{H}$ is a recursive linear order with no hyperarithmetic descending sequences.

$\mathcal{H} \cong \omega_1^{CK} \times (1 + \mathbb{Q})$

Thus, $\mathcal{H}$ provides a notation to each recursive ordinal.

Using Gödel's fixed point lemma, we can iterate *Con* along $\mathcal{H}$.

We say that f *majorizes* g if there is a true φ such that whenever $\psi \vdash \varphi$ then $f(\psi)$ strictly implies $g(\psi)$.

Theorem (Montalbán–W.)

For every non-standard $\alpha \in \mathcal{H}$ and standard $\beta \in \mathcal{H}$,
(i) $\varphi \mapsto (\varphi \wedge Con^{\alpha}(\varphi))$ majorizes $\varphi \mapsto (\varphi \wedge Con^{\beta}(\varphi))$ but
(ii) $\varphi \mapsto (\varphi \wedge 1Con(\varphi)$ majorizes $\varphi \mapsto (\varphi \wedge Con^{\alpha}(\varphi))$.

From *cofinal* to *in the limit*

We would like to strengthen our positive results by changing *cofinally* to *in the limit*.

Let f be recursive and monotonic. Suppose that for all φ
(i) $\varphi \wedge Con(\varphi)$ implies $f(\varphi)$ and
(ii) $f(\varphi)$ implies φ.

Question: Must f be equivalent to the identity or to *Con* on a true ideal?

Question: Is the relation of cofinal agreement on true sentences an equivalence relation on recursive monotonic operators?

Thanks!

L. Beklemishev (2003)
Proof-theoretic analysis by iterated reflection.
Archive for Mathematical Logic. vol. 42, no. 6.

A. Montalbán and J. Walsh (2017)
On the inevitability of the consistency operator.
arXiv.

V. Shavrukov and A. Visser (2014)
Uniform density in Lindenbaum algebras.
Notre Dame Journal of Formal Logic. vol. 55, no. 4.

T. Slaman and J. Steel (1988)
Definable functions on degrees.
Cabal Seminar. 81–85, p. 37–55

A. Visser (1990)
Interpretability logic.
Mathematical Logic. (P. P. Petkov, ed.) Plenum Press. 175–209.