INL (instantial neighborhood logic)
- tableau, sequent calculus, interpolation

Junhua Yu
(junhua.yu.5036@outlook.com)

Tsinghua University (Beijing, China)

2017.10.18 @ Steklov Mathematical Institute

Outline

- Instantial neighborhood logic INL
- Semantic tableau system TABinl (presented at TSU in 2016)
- Sequent calculus G3inl
- Lyndon interpolation theorem
- Future directions

Abbreviation: “nbd” means “neighborhood”

Instantial neighborhood logic INL

Nbd structures

- Frame: $\mathfrak{F} = (W, \sigma)$
  - $W \neq \varnothing$, a domain;
  - $\sigma : W \mapsto 2^{2^W}$, a nbd function.
- Possible properties of nbd functions
  - nbd's are non-empty,
  - nbd's of w always contain w,
  - each point has exactly 1 nbd
    - degenerates to relational semantics,
  - closure properties...

  None of above assumed for the current work.
- Model: $\mathfrak{M} = (\mathfrak{F}, V)$
  - $\mathfrak{F}$, a nbd frame;
  - $V : W \mapsto 2^{\mathcal{P}}$, a propositional valuation.

Nbd structures

- Frame: $\mathfrak{F} = (W, \sigma)$
  - $W \neq \varnothing$, a domain;
  - $\sigma : W \mapsto 2^{2^W}$, a nbd function.
- Possible properties of nbd functions
  - nbd's are non-empty,
  - nbd's of w always contain w,
  - each point has exactly 1 nbd
    - degenerates to relational semantics,
  - closure properties...

  None of above assumed for the current work.
- Model: $\mathfrak{M} = (\mathfrak{F}, V)$
  - $\mathfrak{F}$, a nbd frame;
  - $V : W \mapsto 2^{\mathcal{P}}$, a propositional valuation.

Nbd structures

- Frame: $\mathfrak{F} = (W, \sigma)$
  - $W \neq \varnothing$, a domain;
  - $\sigma : W \mapsto 2^{2^W}$, a nbd function.
- Possible properties of nbd functions
  - nbd's are non-empty,
  - nbd's of w always contain w,
  - each point has exactly 1 nbd
    - degenerates to relational semantics,
  - closure properties...

  None of above assumed for the current work.
- Model: $\mathfrak{M} = (\mathfrak{F}, V)$
  - $\mathfrak{F}$, a nbd frame;
  - $V : W \mapsto 2^{\mathcal{P}}$, a propositional valuation.

Nbd logic

- Basic modal language: unary operator $\Box$.
- Truth definition - a $\exists\forall$ reading of $\Box$:
  - $\mathfrak{M}, w \models \Box\alpha$
    iff
    $(\exists N \in \sigma(w))(\forall n \in N)\ \mathfrak{M}, n \models \alpha$.
  - a neighborhood (of the current point) has α true everywhere inside.
  - B.t.w., alternative truth definition exist.
- Some schemes as examples:

invalid	valid
$\Box\alpha \wedge \Box\beta \to \Box(\alpha \wedge \beta)$	$\Box(\alpha \wedge \beta) \to \Box\alpha \wedge \Box\beta$
$\dfrac{\models \phi}{\models \Box\phi}$	$\Box\bot \to \Box\alpha$

Nbd logic

- Basic modal language: unary operator $\Box$.
- Truth definition - a $\exists\forall$ reading of $\Box$:
  - $\mathfrak{M}, w \models \Box\alpha$
    iff
    $(\exists N \in \sigma(w))(\forall n \in N)\, \mathfrak{M}, n \models \alpha$.
  - a neighborhood (of the current point) has α true everywhere inside.
  - B.t.w., alternative truth definition exist.
- Some schemes as examples:

invalid	valid
$\Box\alpha \wedge \Box\beta \rightarrow \Box(\alpha \wedge \beta)$	$\Box(\alpha \wedge \beta) \rightarrow \Box\alpha \wedge \Box\beta$
$\dfrac{\models \phi}{\models \Box\phi}$	$\Box\bot \rightarrow \Box\alpha$

Nbd logic

- Basic modal language: unary operator $\Box$.
- Truth definition - a $\exists\forall$ reading of $\Box$:
  - $\mathfrak{M}, w \models \Box\alpha$
    iff
    $(\exists N \in \sigma(w))(\forall n \in N)\, \mathfrak{M}, n \models \alpha$.
  - a neighborhood (of the current point) has α true everywhere inside.
  - B.t.w., alternative truth definition exist.
- Some schemes as examples:

invalid	valid
$\Box\alpha \wedge \Box\beta \to \Box(\alpha \wedge \beta)$	$\Box(\alpha \wedge \beta) \to \Box\alpha \wedge \Box\beta$
$\dfrac{\models \phi}{\models \Box\phi}$	$\Box\bot \to \Box\alpha$

Nbd logic

- Basic modal language: unary operator $\Box$.
- Truth definition - a $\exists\forall$ reading of $\Box$:
  - $\mathfrak{M}, w \models \Box\alpha$
    iff
    $(\exists N \in \sigma(w))(\forall n \in N)\, \mathfrak{M}, n \models \alpha$.
  - a neighborhood (of the current point) has α true everywhere inside.
  - B.t.w., alternative truth definition exist.
- Some schemes as examples:

invalid	valid
$\Box\alpha \wedge \Box\beta \to \Box(\alpha \wedge \beta)$	$\Box(\alpha \wedge \beta) \to \Box\alpha \wedge \Box\beta$
$\dfrac{\models \phi}{\models \Box\phi}$	$\Box\bot \to \Box\alpha$

Instantial nbd logic INL

- van Benthem et.al. 2017: *RSL* 10(1).
- Describe nbd structures using an "instantial" language:
  - 2-sorted, $(j; 1)$-ary (where $j \in \mathbb{N}$) operator:

$$\Box(\alpha_1, ..., \alpha_j; \alpha_0)$$

  - Subformulas $\alpha_1, ..., \alpha_j$ are called instances.
- Truth definition - a "$\exists(\overrightarrow{\exists}; \forall)$" reading of $\Box$:
  - $\mathfrak{M}, w \vDash \Box(\alpha_1, ..., \alpha_j; \alpha_0)$ iff

$$(\exists N \in \sigma(w)) \begin{cases} (\forall n \in N)\, \mathfrak{M}, n \vDash \alpha_0 \\ (\exists n_1 \in N)\, \mathfrak{M}, n_1 \vDash \alpha_1 \\ \vdots \\ (\exists n_j \in N)\, \mathfrak{M}, n_j \vDash \alpha_j \end{cases}$$

  - a neighborhood (of the current point) in which
    - α_0 holds everywhere, and
    - each instance (respectively) holds somewhere.

Instantial nbd logic INL

- van Benthem et.al. 2017: *RSL* 10(1).
- Describe nbd structures using an "instantial" language:
  - 2-sorted, $(j;1)$-ary (where $j \in \mathbb{N}$) operator:

$$\Box(\alpha_1, ..., \alpha_j; \alpha_0)$$

  - Subformulas $\alpha_1, ..., \alpha_j$ are called instances.
- Truth definition - a "$\exists(\overrightarrow{\exists}; \forall)$" reading of $\Box$:
  - $\mathfrak{M}, w \vDash \Box(\alpha_1, ..., \alpha_j; \alpha_0)$ iff

$$(\exists N \in \sigma(w)) \begin{cases} (\forall n \in N)\, \mathfrak{M}, n \vDash \alpha_0 \\ (\exists n_1 \in N)\, \mathfrak{M}, n_1 \vDash \alpha_1 \\ \vdots \\ (\exists n_j \in N)\, \mathfrak{M}, n_j \vDash \alpha_j \end{cases}$$

  - a neighborhood (of the current point) in which
    - α_0 holds everywhere, and
    - each instance (respectively) holds somewhere.

Instantial nbd logic INL

- van Benthem et.al. 2017: *RSL* 10(1).
- Describe nbd structures using an “instantial” language:
  - 2-sorted, $(j; 1)$-ary (where $j \in \mathbb{N}$) operator:

$$\Box(\alpha_1, ..., \alpha_j; \alpha_0)$$

  - Subformulas $\alpha_1, ..., \alpha_j$ are called instances.
- Truth definition - a “$\exists(\overrightarrow{\exists}; \forall)$” reading of $\Box$:
  - $\mathfrak{M}, w \models \Box(\alpha_1, ..., \alpha_j; \alpha_0)$ iff

$$(\exists N \in \sigma(w)) \begin{cases} (\forall n \in N)\, \mathfrak{M}, n \models \alpha_0 \\ (\exists n_1 \in N)\, \mathfrak{M}, n_1 \models \alpha_1 \\ \quad\vdots \\ (\exists n_j \in N)\, \mathfrak{M}, n_j \models \alpha_j \end{cases}$$

  - a neighborhood (of the current point) in which
    - α_0 holds everywhere, and
    - each instance (respectively) holds somewhere.

- Some invalid schemes:
  - $\neg\Box(;\bot)$ (cf. $\neg\Box\bot$ in nbd logic)
    - but, $\neg\Box(\alpha;\bot)$ is valid
  - $\dfrac{\models \phi}{\models \Box(;\phi)}$ and also $\Box(;\alpha)\wedge\Box(;\beta)\rightarrow\Box(;\alpha\wedge\beta)$
  - $\Box(\alpha;\psi)\wedge\Box(\beta;\psi)\rightarrow\Box(\alpha,\beta;\psi)$
- Some valid schemes:
  - $\Box(\alpha_1,\ldots,\alpha_j;\alpha_0)\rightarrow\Box(\alpha_1,\ldots,\alpha_j;\alpha_0\vee\eta)$
  - $\Box(\alpha_1,\ldots,\alpha_j,\phi;\alpha_0)\rightarrow\Box(\alpha_1,\ldots,\alpha_j,\phi\vee\psi;\alpha_0)$
  - $\Box(\alpha_1,\alpha_2,\ldots,\alpha_j;\alpha_0)\rightarrow\Box(\alpha_2,\ldots,\alpha_j;\alpha_0)$
  - $\Box(\alpha_1,\ldots,\alpha_j;\alpha_0)\rightarrow\Box(\alpha_1,\ldots,\alpha_j,\alpha_i;\alpha_0)$ where $i\in\{1,\ldots,j\}$
  - $\Box(\alpha_1,\ldots,\alpha_j,\eta;\alpha_0)\rightarrow\Box(\alpha_1,\ldots,\alpha_j,\eta\wedge\alpha_0;\alpha_0)$
  - $\neg\Box(\alpha_1,\ldots,\alpha_j,\bot;\alpha_0)$
  - $\Box(\alpha_1,\ldots,\alpha_j;\alpha_0)\rightarrow(\Box(\alpha_1,\ldots,\alpha_j,\delta;\alpha_0)\vee\Box(\alpha_1,\ldots,\alpha_j;\alpha_0\wedge\neg\delta))$
  - Together with propositional logic and replacement of equivalence, these constitutes Hinl, a sound-and-complete axiomatization of INL.

Instantial nbd logic INL

- Some invalid schemes:
  - $\neg\Box(;\bot)$ (cf. $\neg\Box\bot$ in nbd logic)
    - but, $\neg\Box(\alpha;\bot)$ is valid
  - $\dfrac{\vDash \phi}{\vDash \Box(;\phi)}$ and also $\Box(;\alpha)\wedge\Box(;\beta)\rightarrow\Box(;\alpha\wedge\beta)$
  - $\Box(\alpha;\psi)\wedge\Box(\beta;\psi)\rightarrow\Box(\alpha,\beta;\psi)$
- Some valid schemes:
  - $\Box(\alpha_1,...,\alpha_j;\alpha_0)\rightarrow\Box(\alpha_1,...,\alpha_j;\alpha_0\vee\eta)$
  - $\Box(\alpha_1,...,\alpha_j,\phi;\alpha_0)\rightarrow\Box(\alpha_1,...,\alpha_j,\phi\vee\psi;\alpha_0)$
  - $\Box(\alpha_1,\alpha_2,...,\alpha_j;\alpha_0)\rightarrow\Box(\alpha_2,...,\alpha_j;\alpha_0)$
  - $\Box(\alpha_1,...,\alpha_j;\alpha_0)\rightarrow\Box(\alpha_1,...,\alpha_j,\alpha_i;\alpha_0)$ where $i\in\{1,...,j\}$
  - $\Box(\alpha_1,...,\alpha_j,\eta;\alpha_0)\rightarrow\Box(\alpha_1,...,\alpha_j,\eta\wedge\alpha_0;\alpha_0)$
  - $\neg\Box(\alpha_1,...,\alpha_j,\bot;\alpha_0)$
  - $\Box(\alpha_1,...,\alpha_j;\alpha_0)\rightarrow(\Box(\alpha_1,...,\alpha_j,\delta;\alpha_0)\vee\Box(\alpha_1,...,\alpha_j;\alpha_0\wedge\neg\delta))$
  - Together with propositional logic and replacement of equivalence, these constitutes Hinl, a sound-and-complete axiomatization of INL.

Instantial nbd logic INL

- Some invalid schemes:
  - $\neg\Box(;\bot)$ (cf. $\neg\Box\bot$ in nbd logic)
    - but, $\neg\Box(\alpha;\bot)$ is valid
  - $\dfrac{\vDash \phi}{\vDash \Box(;\phi)}$ and also $\Box(;\alpha)\wedge\Box(;\beta)\to\Box(;\alpha\wedge\beta)$
  - $\Box(\alpha;\psi)\wedge\Box(\beta;\psi)\to\Box(\alpha,\beta;\psi)$
- Some valid schemes:
  - $\Box(\alpha_1,...,\alpha_j;\alpha_0)\to\Box(\alpha_1,...,\alpha_j;\alpha_0\vee\eta)$
  - $\Box(\alpha_1,...,\alpha_j,\phi;\alpha_0)\to\Box(\alpha_1,...,\alpha_j,\phi\vee\psi;\alpha_0)$
  - $\Box(\alpha_1,\alpha_2,...,\alpha_j;\alpha_0)\to\Box(\alpha_2,...,\alpha_j;\alpha_0)$
  - $\Box(\alpha_1,...,\alpha_j;\alpha_0)\to\Box(\alpha_1,...,\alpha_j,\alpha_i;\alpha_0)$ where $i\in\{1,...,j\}$
  - $\Box(\alpha_1,...,\alpha_j,\eta;\alpha_0)\to\Box(\alpha_1,...,\alpha_j,\eta\wedge\alpha_0;\alpha_0)$
  - $\neg\Box(\alpha_1,...,\alpha_j,\bot;\alpha_0)$
  - $\Box(\alpha_1,...,\alpha_j;\alpha_0)\to(\Box(\alpha_1,...,\alpha_j,\delta;\alpha_0)\vee\Box(\alpha_1,...,\alpha_j;\alpha_0\wedge\neg\delta))$
  - Together with propositional logic and replacement of equivalence, these constitutes Hinl, a sound-and-complete axiomatization of INL.

Instantial nbd logic INL

- Some invalid schemes:
  - $\neg\Box(;\bot)$ (cf. $\neg\Box\bot$ in nbd logic)
    - but, $\neg\Box(\alpha;\bot)$ is valid
  - $\dfrac{\models\phi}{\models\Box(;\phi)}$ and also $\Box(;\alpha)\land\Box(;\beta)\to\Box(;\alpha\land\beta)$
  - $\Box(\alpha;\psi)\land\Box(\beta;\psi)\to\Box(\alpha,\beta;\psi)$
- Some valid schemes:
  - $\Box(\alpha_1,...,\alpha_j;\alpha_0)\to\Box(\alpha_1,...,\alpha_j;\alpha_0\lor\eta)$
  - $\Box(\alpha_1,...,\alpha_j,\phi;\alpha_0)\to\Box(\alpha_1,...,\alpha_j,\phi\lor\psi;\alpha_0)$
  - $\Box(\alpha_1,\alpha_2,...,\alpha_j;\alpha_0)\to\Box(\alpha_2,...,\alpha_j;\alpha_0)$
  - $\Box(\alpha_1,...,\alpha_j;\alpha_0)\to\Box(\alpha_1,...,\alpha_j,\alpha_i;\alpha_0)$ where $i\in\{1,...,j\}$
  - $\Box(\alpha_1,...,\alpha_j,\eta;\alpha_0)\to\Box(\alpha_1,...,\alpha_j,\eta\land\alpha_0;\alpha_0)$
  - $\neg\Box(\alpha_1,...,\alpha_j,\bot;\alpha_0)$
  - $\Box(\alpha_1,...,\alpha_j;\alpha_0)\to(\Box(\alpha_1,...,\alpha_j,\delta;\alpha_0)\lor\Box(\alpha_1,...,\alpha_j;\alpha_0\land\neg\delta))$
  - Together with propositional logic and replacement of equivalence, these constitutes Hinl, a sound-and-complete axiomatization of INL.

Instantial nbd logic INL

- Some invalid schemes:
  - $\neg\Box(;\bot)$ (cf. $\neg\Box\bot$ in nbd logic)
    - but, $\neg\Box(\alpha;\bot)$ is valid
  - $\dfrac{\models\phi}{\models\Box(;\phi)}$ and also $\Box(;\alpha)\wedge\Box(;\beta)\rightarrow\Box(;\alpha\wedge\beta)$
  - $\Box(\alpha;\psi)\wedge\Box(\beta;\psi)\rightarrow\Box(\alpha,\beta;\psi)$
- Some valid schemes:
  - $\Box(\alpha_1,...,\alpha_j;\alpha_0)\rightarrow\Box(\alpha_1,...,\alpha_j;\alpha_0\vee\eta)$
  - $\Box(\alpha_1,...,\alpha_j,\phi;\alpha_0)\rightarrow\Box(\alpha_1,...,\alpha_j,\phi\vee\psi;\alpha_0)$
  - $\Box(\alpha_1,\alpha_2,...,\alpha_j;\alpha_0)\rightarrow\Box(\alpha_2,...,\alpha_j;\alpha_0)$
  - $\Box(\alpha_1,...,\alpha_j;\alpha_0)\rightarrow\Box(\alpha_1,...,\alpha_j,\alpha_i;\alpha_0)$ where $i\in\{1,...,j\}$
  - $\Box(\alpha_1,...,\alpha_j,\eta;\alpha_0)\rightarrow\Box(\alpha_1,...,\alpha_j,\eta\wedge\alpha_0;\alpha_0)$
  - $\neg\Box(\alpha_1,...,\alpha_j,\bot;\alpha_0)$
  - $\Box(\alpha_1,...,\alpha_j;\alpha_0)\rightarrow(\Box(\alpha_1,...,\alpha_j,\delta;\alpha_0)\vee\Box(\alpha_1,...,\alpha_j;\alpha_0\wedge\neg\delta))$
  - Together with propositional logic and replacement of equivalence, these constitutes Hinl, a sound-and-complete axiomatization of INL.

Instantial nbd logic INL

- Some invalid schemes:
  - $\neg\Box(;\bot)$ (cf. $\neg\Box\bot$ in nbd logic)
    - but, $\neg\Box(\alpha;\bot)$ is valid
  - $\dfrac{\models \phi}{\models \Box(;\phi)}$ and also $\Box(;\alpha)\wedge\Box(;\beta)\rightarrow\Box(;\alpha\wedge\beta)$
  - $\Box(\alpha;\psi)\wedge\Box(\beta;\psi)\rightarrow\Box(\alpha,\beta;\psi)$
- Some valid schemes:
  - $\Box(\alpha_1,...,\alpha_j;\alpha_0)\rightarrow\Box(\alpha_1,...,\alpha_j;\alpha_0\vee\eta)$
  - $\Box(\alpha_1,...,\alpha_j,\phi;\alpha_0)\rightarrow\Box(\alpha_1,...,\alpha_j,\phi\vee\psi;\alpha_0)$
  - $\Box(\alpha_1,\alpha_2,...,\alpha_j;\alpha_0)\rightarrow\Box(\alpha_2,...,\alpha_j;\alpha_0)$
  - $\Box(\alpha_1,...,\alpha_j;\alpha_0)\rightarrow\Box(\alpha_1,...,\alpha_j,\alpha_i;\alpha_0)$ where $i\in\{1,...,j\}$
  - $\Box(\alpha_1,...,\alpha_j,\eta;\alpha_0)\rightarrow\Box(\alpha_1,...,\alpha_j,\eta\wedge\alpha_0;\alpha_0)$
  - $\neg\Box(\alpha_1,...,\alpha_j,\bot;\alpha_0)$
  - $\Box(\alpha_1,...,\alpha_j;\alpha_0)\rightarrow(\Box(\alpha_1,...,\alpha_j,\delta;\alpha_0)\vee\Box(\alpha_1,...,\alpha_j;\alpha_0\wedge\neg\delta))$
  - Together with propositional logic and replacement of equivalence, these constitutes Hinl, a sound-and-complete axiomatization of INL.

Instantial nbd logic INL

- Some invalid schemes:
  - $\neg\Box(;\bot)$ (cf. $\neg\Box\bot$ in nbd logic)
    - but, $\neg\Box(\alpha;\bot)$ is valid
  - $\dfrac{\vDash \phi}{\vDash \Box(;\phi)}$ and also $\Box(;\alpha)\wedge\Box(;\beta)\to\Box(;\alpha\wedge\beta)$
  - $\Box(\alpha;\psi)\wedge\Box(\beta;\psi)\to\Box(\alpha,\beta;\psi)$
- Some valid schemes:
  - $\Box(\alpha_1,...,\alpha_j;\alpha_0)\to\Box(\alpha_1,...,\alpha_j;\alpha_0\vee\eta)$
  - $\Box(\alpha_1,...,\alpha_j,\phi;\alpha_0)\to\Box(\alpha_1,...,\alpha_j,\phi\vee\psi;\alpha_0)$
  - $\Box(\alpha_1,\alpha_2,...,\alpha_j;\alpha_0)\to\Box(\alpha_2,...,\alpha_j;\alpha_0)$
  - $\Box(\alpha_1,...,\alpha_j;\alpha_0)\to\Box(\alpha_1,...,\alpha_j,\alpha_i;\alpha_0)$ where $i\in\{1,...,j\}$
  - $\Box(\alpha_1,...,\alpha_j,\eta;\alpha_0)\to\Box(\alpha_1,...,\alpha_j,\eta\wedge\alpha_0;\alpha_0)$
  - $\neg\Box(\alpha_1,...,\alpha_j,\bot;\alpha_0)$
  - $\Box(\alpha_1,...,\alpha_j;\alpha_0)\to(\Box(\alpha_1,...,\alpha_j,\delta;\alpha_0)\vee\Box(\alpha_1,...,\alpha_j;\alpha_0\wedge\neg\delta))$
  - Together with propositional logic and replacement of equivalence, these constitutes Hinl, a sound-and-complete axiomatization of INL.

Instantial nbd logic INL

- Some invalid schemes:
  - $\neg\Box(;\bot)$ (cf. $\neg\Box\bot$ in nbd logic)
    - but, $\neg\Box(\alpha;\bot)$ is valid
  - $\dfrac{\vDash\phi}{\vDash\Box(;\phi)}$ and also $\Box(;\alpha)\wedge\Box(;\beta)\rightarrow\Box(;\alpha\wedge\beta)$
  - $\Box(\alpha;\psi)\wedge\Box(\beta;\psi)\rightarrow\Box(\alpha,\beta;\psi)$
- Some valid schemes:
  - $\Box(\alpha_1,...,\alpha_j;\alpha_0)\rightarrow\Box(\alpha_1,...,\alpha_j;\alpha_0\vee\eta)$
  - $\Box(\alpha_1,...,\alpha_j,\phi;\alpha_0)\rightarrow\Box(\alpha_1,...,\alpha_j,\phi\vee\psi;\alpha_0)$
  - $\Box(\alpha_1,\alpha_2,...,\alpha_j;\alpha_0)\rightarrow\Box(\alpha_2,...,\alpha_j;\alpha_0)$
  - $\Box(\alpha_1,...,\alpha_j;\alpha_0)\rightarrow\Box(\alpha_1,...,\alpha_j,\alpha_i;\alpha_0)$ where $i\in\{1,...,j\}$
  - $\Box(\alpha_1,...,\alpha_j,\eta;\alpha_0)\rightarrow\Box(\alpha_1,...,\alpha_j,\eta\wedge\alpha_0;\alpha_0)$
  - $\neg\Box(\alpha_1,...,\alpha_j,\bot;\alpha_0)$
  - $\Box(\alpha_1,...,\alpha_j;\alpha_0)\rightarrow(\Box(\alpha_1,...,\alpha_j,\delta;\alpha_0)\vee\Box(\alpha_1,...,\alpha_j;\alpha_0\wedge\neg\delta))$
  - Together with propositional logic and replacement of equivalence, these constitutes Hinl, a sound-and-complete axiomatization of INL.

Semantic tableau system TABinl

Semantic tableau system TABinl

- General idea: satisfiability reduction
  - Start with the main goal of satisfying the negation;
  - Reduce goals to subgoals (by rules), extend potential model upon request;
  - Impossible goals are "closed", otherwise "open" (hints of models).
- Rules for classical propositional logic

  ||...|| means branching

$$\frac{\neg\neg\phi}{\phi} \quad \frac{\alpha \wedge \beta}{\begin{matrix}\alpha \\ \beta\end{matrix}} \quad \frac{\neg(\alpha \vee \beta)}{\begin{matrix}\neg\alpha \\ \neg\beta\end{matrix}} \quad \frac{\neg(\alpha \rightarrow \beta)}{\begin{matrix}\alpha \\ \neg\beta\end{matrix}} \quad \frac{\neg(\alpha \wedge \beta)}{||\neg\alpha\,||\,\neg\beta||} \quad \frac{\alpha \vee \beta}{||\alpha\,||\,\beta||} \quad \frac{\alpha \rightarrow \beta}{||\neg\alpha\,||\,\beta||}$$

- Still missing for INL are rule(s) for nbd operator $\Box$.

Semantic tableau system TABinl

- General idea: satisfiability reduction
  - Start with the main goal of satisfying the negation;
  - Reduce goals to subgoals (by rules), extend potential model upon request;
  - Impossible goals are "closed", otherwise "open" (hints of models).
- Rules for classical propositional logic

  ||...|| means branching

$$\frac{\neg\neg\phi}{\phi} \quad \frac{\alpha\wedge\beta}{\begin{matrix}\alpha\\ \beta\end{matrix}} \quad \frac{\neg(\alpha\vee\beta)}{\begin{matrix}\neg\alpha\\ \neg\beta\end{matrix}} \quad \frac{\neg(\alpha\rightarrow\beta)}{\begin{matrix}\alpha\\ \neg\beta\end{matrix}} \quad \frac{\neg(\alpha\wedge\beta)}{||\neg\alpha\,||\,\neg\beta||} \quad \frac{\alpha\vee\beta}{||\alpha\,||\,\beta||} \quad \frac{\alpha\rightarrow\beta}{||\neg\alpha\,||\,\beta||}$$

- Still missing for INL are rule(s) for nbd operator $\Box$.

Semantic tableau system TABinl

- General idea: satisfiability reduction
  - Start with the main goal of satisfying the negation;
  - Reduce goals to subgoals (by rules), extend potential model upon request;
  - Impossible goals are "closed", otherwise "open" (hints of models).
- Rules for classical propositional logic

  ||...|| means branching

$$\frac{\neg\neg\phi}{\phi} \qquad \frac{\alpha \wedge \beta}{\begin{array}{c}\alpha\\ \beta\end{array}} \qquad \frac{\neg(\alpha \vee \beta)}{\begin{array}{c}\neg\alpha\\ \neg\beta\end{array}} \qquad \frac{\neg(\alpha \rightarrow \beta)}{\begin{array}{c}\alpha\\ \neg\beta\end{array}} \qquad \frac{\neg(\alpha \wedge \beta)}{||\neg\alpha \,||\, \neg\beta||} \qquad \frac{\alpha \vee \beta}{||\alpha \,||\, \beta||} \qquad \frac{\alpha \rightarrow \beta}{||\neg\alpha \,||\, \beta||}$$

- Still missing for INL are rule(s) for nbd operator $\Box$.

Semantic tableau system TABinl

- General idea: satisfiability reduction
  - Start with the main goal of satisfying the negation;
  - Reduce goals to subgoals (by rules),
    extend potential model upon request;
  - Impossible goals are "closed",
    otherwise "open" (hints of models).
- Rules for classical propositional logic

  $||\ldots||$ means branching

$$\frac{\neg\neg\phi}{\phi} \qquad \frac{\alpha \wedge \beta}{\begin{array}{c}\alpha\\ \beta\end{array}} \qquad \frac{\neg(\alpha \vee \beta)}{\begin{array}{c}\neg\alpha\\ \neg\beta\end{array}} \qquad \frac{\neg(\alpha \rightarrow \beta)}{\begin{array}{c}\alpha\\ \neg\beta\end{array}} \qquad \frac{\neg(\alpha \wedge \beta)}{||\neg\alpha \,||\, \neg\beta||} \qquad \frac{\alpha \vee \beta}{||\alpha \,||\, \beta||} \qquad \frac{\alpha \rightarrow \beta}{||\neg\alpha \,||\, \beta||}$$

- Still missing for INL are rule(s) for nbd operator $\Box$.

Semantic tableau system TABinl

- General idea: satisfiability reduction
  - Start with the main goal of satisfying the negation;
  - Reduce goals to subgoals (by rules),
    extend potential model upon request;
  - Impossible goals are "closed",
    otherwise "open" (hints of models).
- Rules for classical propositional logic

  $||\ldots||$ means branching

$$\frac{\neg\neg\phi}{\phi} \qquad \frac{\alpha\wedge\beta}{\begin{array}{c}\alpha\\ \beta\end{array}} \qquad \frac{\neg(\alpha\vee\beta)}{\begin{array}{c}\neg\alpha\\ \neg\beta\end{array}} \qquad \frac{\neg(\alpha\rightarrow\beta)}{\begin{array}{c}\alpha\\ \neg\beta\end{array}} \qquad \frac{\neg(\alpha\wedge\beta)}{||\neg\alpha\,||\,\neg\beta||} \qquad \frac{\alpha\vee\beta}{||\alpha\,||\,\beta||} \qquad \frac{\alpha\rightarrow\beta}{||\neg\alpha\,||\,\beta||}$$

- Still missing for INL are rule(s) for nbd operator $\Box$.

Semantic tableau system TABinl

- In order to be satisfied:
  - A $\Box$-formula requires a nbd (of certain type);
    A $\neg\Box$-formula refutes any nbd (of certain type);
    Other formulas govern locally.
  - $\Box$-formulas do not work together to close a goal; they each does,
    together with all $\neg\Box$-formulas in the same goal.
- The rule takes from a goal:
  - one $\Box$-formula (with j-many instances), and
  - any finite number (say k) of $\neg\Box$-formulas
    (resp. with $j_1, ..., j_k$-many instances).

$$\begin{array}{c}\Box(\alpha_1, ..., \alpha_j; \alpha_0)\\ \neg\Box(\beta^1_1, ..., \beta^1_{j_1}; \beta^1_0)\\ \vdots\\ \neg\Box(\beta^k_1, ..., \beta^k_{j_k}; \beta^k_0)\\ \hline \end{array}$$

Semantic tableau system TABinl

- In order to be satisfied:
  - A $\Box$-formula requires a nbd (of certain type);
    A $\neg\Box$-formula refutes any nbd (of certain type);
    Other formulas govern locally.
  - $\Box$-formulas do not work together to close a goal;
    they each does,
    together with all $\neg\Box$-formulas in the same goal.
- The rule takes from a goal:
  - one $\Box$-formula (with j-many instances), and
  - any finite number (say k) of $\neg\Box$-formulas
    (resp. with $j_1, \ldots, j_k$-many instances).

$$\begin{array}{c} \Box(\alpha_1, \ldots, \alpha_j; \alpha_0) \\ \neg\Box(\beta^1_1, \ldots, \beta^1_{j_1}; \beta^1_0) \\ \vdots \\ \neg\Box(\beta^k_1, \ldots, \beta^k_{j_k}; \beta^k_0) \\ \hline \end{array}$$

Semantic tableau system TABinl

- In order to be satisfied:
  - A $\Box$-formula requires a nbd (of certain type);
    A $\neg\Box$-formula refutes any nbd (of certain type);
    Other formulas govern locally.
  - $\Box$-formulas do not work together to close a goal;
    they each does,
    together with all $\neg\Box$-formulas in the same goal.
- The rule takes from a goal:
  - one $\Box$-formula (with j-many instances), and
  - any finite number (say k) of $\neg\Box$-formulas
    (resp. with $j_1, ..., j_k$-many instances).

$$\frac{\begin{array}{c}\Box(\alpha_1, ..., \alpha_j; \alpha_0)\\ \neg\Box(\beta^1_1, ..., \beta^1_{j_1}; \beta^1_0)\\ \vdots\\ \neg\Box(\beta^k_1, ..., \beta^k_{j_k}; \beta^k_0)\end{array}}{}$$

Semantic tableau system TABinl

$$\frac{\begin{array}{c}\Box(\alpha_1, ..., \alpha_j; \alpha_0)\\ \neg\Box(\beta^1_1, ..., \beta^1_{j_1}; \beta^1_0)\\ \vdots \\ \neg\Box(\beta^k_1, ..., \beta^k_{j_k}; \beta^k_0)\end{array}}{\Big\| \ \big| \alpha_0 \wedge \sigma \qquad \big|_{\sigma \in \{\alpha_x\}^j_{x=1}} \ \Big\|}$$

- $\Box(\alpha_1, ..., \alpha_j; \alpha_0)$ requires a nbd with (generally) j points.
  A nbd is consistent, if all its points are.
- $\forall i \in \{1, ..., k\}$, $\neg\Box(\beta^i_1, ..., \beta^i_{j_i}; \beta^i_0)$ requires
  either - β^i_0 fails at some point,
  or - β^i_h fails at each point for some $h \in \{1, ..., j_i\}$.
- $j_i + 1$ options for each i, so $\prod^k_{z=1}(j_z + 1)$ options in total.
  Index possible nbd's by the option it takes, e.g., $I = \langle I(1), ..., I(k)\rangle$.

Semantic tableau system TABinl

$$\frac{\begin{array}{c}\Box(\alpha_1, ..., \alpha_j; \alpha_0) \\ \neg\Box(\beta^1_1, ..., \beta^1_{j_1}; \beta^1_0) \\ \vdots \\ \neg\Box(\beta^k_1, ..., \beta^k_{j_k}; \beta^k_0)\end{array}}{\Big\| \; \big|\alpha_0 \wedge \sigma \qquad \big|_{\sigma \in \{\alpha_x\}^j_{x=1} \cup \{\neg\beta^i_0\}} \; \Big\|}$$

- $\Box(\alpha_1, ..., \alpha_j; \alpha_0)$ requires a nbd with (generally) j points.

  A nbd is consistent, if all its points are.
- $\forall i \in \{1, ..., k\}$, $\neg\Box(\beta^i_1, ..., \beta^i_{j_i}; \beta^i_0)$ requires

  either - β^i_0 fails at some point,

  or - β^i_h fails at each point for some $h \in \{1, ..., j_i\}$.
- $j_i + 1$ options for each i, so $\prod^k_{z=1}(j_z + 1)$ options in total.

  Index possible nbd's by the option it takes, e.g., $I = \langle I(1), ..., I(k)\rangle$.

Semantic tableau system TABinl

$$\frac{\begin{array}{c}\Box(\alpha_1, ..., \alpha_j; \alpha_0)\\ \neg\Box(\beta^1_1, ..., \beta^1_{j_1}; \beta^1_0)\\ \vdots \\ \neg\Box(\beta^k_1, ..., \beta^k_{j_k}; \beta^k_0)\end{array}}{\Big|\Big|\ |\alpha_0 \wedge \sigma \wedge \qquad \neg\beta^i{}_h|_{\sigma\in\{\alpha_x\}^j_{x=1}\cup\{\neg\beta^i_0\}}\ \Big|\Big|}$$

- $\Box(\alpha_1, ..., \alpha_j; \alpha_0)$ requires a nbd with (generally) j points.
  A nbd is consistent, if all its points are.
- $\forall i \in \{1, ..., k\}$, $\neg\Box(\beta^i_1, ..., \beta^i_{j_i}; \beta^i_0)$ requires that
  either - β^i_0 fails at some point,
  or - β^i_h fails at each point for some $h \in \{1, ..., j_i\}$.
- $j_i + 1$ options for each i, so $\prod_{z=1}^{k}(j_z + 1)$ options in total.
  Index possible nbd's by the option it takes, e.g., $I = \langle I(1), ..., I(k)\rangle$.

$$\frac{\begin{array}{c}\Box(\alpha_1, ..., \alpha_j; \alpha_0)\\ \neg\Box(\beta^1_1, ..., \beta^1_{j_1}; \beta^1_0)\\ \vdots \\ \neg\Box(\beta^k_1, ..., \beta^k_{j_k}; \beta^k_0)\end{array}}{\Big\|\, \big|\alpha_0 \wedge \sigma \wedge \bigwedge^{I(i)\neq 0}_{i\in\{1,...,k\}} \neg\beta^i_{I(i)}\big|_{\sigma\in\{\alpha_x\}^j_{x=1}\cup\{\neg\beta^y_0\}^{I(y)=0}_{y\in\{1,...,k\}}} \,\Big\|_{I\in\bigotimes^k_{z=1}\{0,...,j_z\}}}$$

- $\Box(\alpha_1, ..., \alpha_j; \alpha_0)$ requires a nbd with (generally) j points.

  A nbd is consistent, if all its points are.
- $\forall i \in \{1, ..., k\}$, $\neg\Box(\beta^i_1, ..., \beta^i_{j_i}; \beta^i_0)$ requires that

  either - β^i_0 fails at some point - "$I(i) = 0$"

  or - β^i_h fails at each point for some $h \in \{1, ..., j_i\}$ - "$I(i) \neq 0$".
- $j_i + 1$ options for each i, so $\prod^k_{z=1}(j_z + 1)$ options in total.

  Index possible nbd's by the option it takes, e.g., $I = \langle I(1), ..., I(k)\rangle$.

Semantic tableau system TABinl

$$\frac{\begin{array}{c}\Box(\alpha_1,\dots,\alpha_j;\alpha_0)\\ \neg\Box(\beta^1_1,\dots,\beta^1_{j_1};\beta^1_0)\\ \vdots\\ \neg\Box(\beta^k_1,\dots,\beta^k_{j_k};\beta^k_0)\end{array}}{\Big\|\,\big|\alpha_0\wedge\sigma\wedge\bigwedge^{l(i)\neq 0}_{i\in\{1,\dots,k\}}\neg\beta^i_{l(i)}\big|_{\sigma\in\{\alpha_x\}^j_{x=1}\cup\{\neg\beta^y_0\}^{l(y)=0}_{y\in\{1,\dots,k\}}}\Big\|_{l\in\bigotimes^k_{z=1}\{0,\dots,j_z\}}}$$

- Each of these options a possible nbd to satisfy the goal,

  In order to close, each option has to be closed.
- Each possible nbd is a collection of points (subgoals),

  To close a nbd, it is enough to close one subgoal in it.
- It is destructive,

  No formulas (used or not) above the line are effective below the line.
- It degenerates to special cases where certain parameters are 0.

Semantic tableau system TABinl

$$\frac{\begin{array}{c}\Box(\alpha_1, ..., \alpha_j; \alpha_0)\\ \neg\Box(\beta^1_1, ..., \beta^1_{j_1}; \beta^1_0)\\ \vdots \\ \neg\Box(\beta^k_1, ..., \beta^k_{j_k}; \beta^k_0)\end{array}}{\Big\| \big| \alpha_0 \wedge \sigma \wedge \bigwedge^{I(i)\neq 0}_{i\in\{1,...,k\}} \neg\beta^i_{I(i)} \big|_{\sigma\in\{\alpha_x\}^j_{x=1}\cup\{\neg\beta^y_0\}^{I(y)=0}_{y\in\{1,...,k\}}} \Big\|_{I\in\bigotimes^k_{z=1}\{0,...,j_z\}}}$$

- Each of these options a possible nbd to satisfy the goal,

  In order to close, each option has to be closed.
- Each possible nbd is a collection of points (subgoals),

  To close a nbd, it is enough to close one subgoal in it.
- It is destructive,

  No formulas (used or not) above the line are effective below the line.
- It degenerates to special cases where certain parameters are 0.

Semantic tableau system TABinl

$$\frac{\begin{array}{c}\Box(\alpha_1, ..., \alpha_j; \alpha_0)\\ \neg\Box(\beta^1_1, ..., \beta^1_{j_1}; \beta^1_0)\\ \vdots \\ \neg\Box(\beta^k_1, ..., \beta^k_{j_k}; \beta^k_0)\end{array}}{\left\|\; \left|\alpha_0 \wedge \sigma \wedge \bigwedge_{i\in\{1,\ldots,k\}}^{l(i)\neq 0} \neg\beta^i_{l(i)}\right|_{\sigma\in\{\alpha_x\}_{x=1}^{j}\cup\{\neg\beta^y_0\}_{y\in\{1,\ldots,k\}}^{l(y)=0}} \;\right\|_{l\in\bigotimes_{z=1}^{k}\{0,\ldots,j_z\}}}$$

- Each of these options a possible nbd to satisfy the goal,

  In order to close, each option has to be closed.

- Each possible nbd is a collection of points (subgoals),

  To close a nbd, it is enough to close one subgoal in it.

- It is destructive,

  No formulas (used or not) above the line are effective below the line.

- It degenerates to special cases where certain parameters are 0.

Semantic tableau system TABinl

$$\frac{\begin{array}{c}\Box(\alpha_1, ..., \alpha_j; \alpha_0)\\ \neg\Box(\beta^1_1, ..., \beta^1_{j_1}; \beta^1_0)\\ \vdots \\ \neg\Box(\beta^k_1, ..., \beta^k_{j_k}; \beta^k_0)\end{array}}{\Big|\Big| \, \big|\alpha_0 \wedge \sigma \wedge \bigwedge_{i\in\{1,...,k\}}^{l(i)\neq 0} \neg\beta^i_{l(i)}\big|_{\sigma\in\{\alpha_x\}^j_{x=1}\cup\{\neg\beta^y_0\}^{l(y)=0}_{y\in\{1,...,k\}}} \, \Big|\Big|_{l\in\bigotimes^k_{z=1}\{0,...,j_z\}}}$$

- Each of these options a possible nbd to satisfy the goal,

  In order to close, each option has to be closed.

- Each possible nbd is a collection of points (subgoals),

  To close a nbd, it is enough to close one subgoal in it.

- It is destructive,

  No formulas (used or not) above the line are effective below the line.

- It degenerates to special cases where certain parameters are 0.

Semantic tableau system TABinl

- TABinl is sound and complete.
- TABinl offers to INL a mechanical procedure of proof/counter-model search.
  - B.t.w., known from van Benthem et.al. 2017: *RSL* 10(1), INL is PSPACE-complete.
- A manuscript introducing TABinl will be available in the proceeding of LFCS 2018.

Semantic tableau system TABinl

- TABinl is sound and complete.
- TABinl offers to INL a mechanical procedure of proof/counter-model search.
  - B.t.w., known from van Benthem et.al. 2017: *RSL* 10(1), INL is PSPACE-complete.
- A manuscript introducing TABinl will be available in the proceeding of LFCS 2018.

Semantic tableau system TABinl

- TABinl is sound and complete.
- TABinl offers to INL a mechanical procedure of proof/counter-model search.
  - B.t.w., known from van Benthem et.al. 2017: *RSL* 10(1), INL is PSPACE-complete.
- A manuscript introducing TABinl will be available in the proceeding of LFCS 2018.

Sequent calculus G3inl

A duality

$$\frac{\begin{array}{c}\Box(\alpha_1, ..., \alpha_j; \alpha_0) \\ \neg\Box(\beta^1_1, ..., \beta^1_{j_1}; \beta^1_0) \\ \vdots \\ \neg\Box(\beta^k_1, ..., \beta^k_{j_k}; \beta^k_0)\end{array}}{\left\| \left| \alpha_0 \wedge \sigma \wedge \bigwedge^{I(i)\neq 0}_{i\in\{1,\dots,k\}} \neg\beta^i_{I(i)} \right|_{\sigma\in\{\alpha_x\}^j_{x=1}\cup\{\neg\beta^y_0\}^{I(y)=0}_{y\in\{1,\dots,k\}}} \right\|_{I\in\bigotimes^k_{z=1}\{0,\dots,j_z\}}}$$

- Sequent/tableau proofs are usually dual of each other.
  - Sequents ⇝ branches where closure is tested.
  - (Left/right) sides to "⇒" ⇝ (positive/negtive) signs of formulas.
- In the nbd rule of TABinl:
  - a goal is reduced to groups (instead of a group) of subgoals,
  - it is sufficient to close one subgoal in each group.

A duality

$$\dfrac{\begin{array}{c}\Box(\alpha_1, \ldots, \alpha_j; \alpha_0)\\ \neg\Box(\beta^1_1, \ldots, \beta^1_{j_1}; \beta^1_0)\\ \vdots \\ \neg\Box(\beta^k_1, \ldots, \beta^k_{j_k}; \beta^k_0)\end{array}}{\left\| \left| \alpha_0 \wedge \sigma \wedge \bigwedge_{i\in\{1,\ldots,k\}}^{I(i)\neq 0} \neg\beta^i_{I(i)} \right|_{\sigma\in\{\alpha_x\}_{x=1}^{j}\cup\{\neg\beta^y_0\}_{y\in\{1,\ldots,k\}}^{I(y)=0}} \right\|_{I\in\bigotimes_{z=1}^{k}\{0,\ldots,j_z\}}}$$

- Sequent/tableau proofs are usually dual of each other.
  - Sequents $\leftrightsquigarrow$ branches where closure is tested.
  - (Left/right) sides to "$\Rightarrow$" $\leftrightsquigarrow$ (positive/negtive) signs of formulas.
- In the nbd rule of TABinl:
  - a goal is reduced to groups (instead of a group) of subgoals,
  - it is sufficient to close one subgoal in each group.

A duality

$$\frac{\begin{array}{c}\Box(\alpha_1, ..., \alpha_j; \alpha_0)\\ \neg\Box(\beta^1_1, ..., \beta^1_{j_1}; \beta^1_0)\\ \vdots \\ \neg\Box(\beta^k_1, ..., \beta^k_{j_k}; \beta^k_0)\end{array}}{\Big|\Big| \, \big|\alpha_0 \wedge \sigma \wedge \bigwedge_{i\in\{1,\dots,k\}}^{I(i)\neq 0} \neg\beta^i_{I(i)}\big|_{\sigma\in\{\alpha_x\}^j_{x=1}\cup\{\neg\beta^y_0\}^{I(y)=0}_{y\in\{1,\dots,k\}}} \Big|\Big|_{I\in\bigotimes^k_{z=1}\{0,\dots,j_z\}}}$$

- Sequent/tableau proofs are usually dual of each other.
  - Sequents $\leftrightsquigarrow$ branches where closure is tested.
  - (Left/right) sides to "$\Rightarrow$" $\leftrightsquigarrow$ (positive/negtive) signs of formulas.
- In the nbd rule of TABinl:
  - a goal is reduced to groups (instead of a group) of subgoals,
  - it is sufficient to close one subgoal in each group.

A duality

$$\frac{\begin{array}{c}\Box(\alpha_1, ..., \alpha_j; \alpha_0) \\ \neg\Box(\beta_1^1, ..., \beta_{j_1}^1; \beta_0^1) \\ \vdots \\ \neg\Box(\beta_1^k, ..., \beta_{j_k}^k; \beta_0^k)\end{array}}{\left|\left| \; \left|\alpha_0 \wedge \sigma \wedge \bigwedge_{i\in\{1,\ldots,k\}}^{l(i)\neq 0} \neg\beta^i_{l(i)}\right|_{\sigma\in\{\alpha_x\}_{x=1}^j \cup \{\neg\beta_0^y\}_{y\in\{1,\ldots,k\}}^{l(y)=0}} \; \right|\right|_{l\in\bigotimes_{z=1}^k\{0,\ldots,j_z\}}}$$

- The dual is a hyper-sequent rule.
  - Hyper-sequents := finite multi-sets of (regular) sequents;
  - like $|\Gamma_1 \Rightarrow \Delta_1|...|\Gamma_n \Rightarrow \Delta_n|$,
    where "|" is read disjunctively.
  - In order to prove a hyper-seq.,
    it is sufficient to prove one of its sequents.

The hyper-sequent rule

$$\frac{\begin{array}{c}\Box(\alpha_1, ..., \alpha_j; \alpha_0) \\ \neg\Box(\beta^1_1, ..., \beta^1_{j_1}; \beta^1_0) \\ \vdots \\ \neg\Box(\beta^k_1, ..., \beta^k_{j_k}; \beta^k_0)\end{array}}{\left|\left| \left|\alpha_0 \wedge \sigma \wedge \bigwedge_{i\in\{1,...,k\}}^{l(i)\neq 0} \neg\beta^i_{l(i)}\right|_{\sigma\in\{\alpha_x\}^j_{x=1}\cup\{\neg\beta^y_0\}^{l(y)=0}_{y\in\{1,...,k\}}} \right|\right|_{l\in\bigotimes^k_{z=1}\{0,...,j_z\}}}$$

$$\frac{\left[\begin{array}{c}\left|\alpha_0, \alpha_{-x} \Rightarrow \left\{\beta^i_{l(i)}\right\}^{l(i)\neq 0}_{i\in\{1,...,k\}}\right|_{x\in\{-j,...,-1\}} \\ \left|\alpha_0 \Rightarrow \beta^y_0, \left\{\beta^i_{l(i)}\right\}^{l(i)\neq 0}_{i\in\{1,...,k\}}\right|^{l(y)=0}_{y\in\{1,...,k\}}\end{array}\right]_{l\in\bigotimes^k_{i=1}\{0,1,...,j_i\}}}{\Box(\alpha_1, ..., \alpha_j; \alpha_0) \Rightarrow \{\Box(\beta^i_1, ..., \beta^i_{j_i}; \beta^i_0)\}^k_{i=1}}$$

The hyper-sequent rule

- For a better appearance, let

$$\left(\begin{array}{lll} J = \{-j, ..., -1\} & K = \{1, ..., k\} & \\ K^{(l)} = \{y \in K \mid l(y) = 0\} & \Omega_K^l = \left\{\beta^i_{l(i)}\right\}_{i \in K}^{l(i) \neq 0} & D = \bigotimes_{i \in K} \{0, 1, ..., j_i\} \end{array}\right)$$

$$\frac{\left[\begin{array}{l} \left| \alpha_0, \alpha_{-x} \Rightarrow \left\{\beta^i_{l(i)}\right\}_{i \in \{1,...,k\}}^{l(i) \neq 0} \right|_{x \in \{-j,...,-1\}} \\ \\ \left| \alpha_0 \Rightarrow \beta^y_0, \left\{\beta^i_{l(i)}\right\}_{i \in \{1,...,k\}}^{l(i) \neq 0} \right|_{y \in \{1,...,k\}}^{l(y)=0} \end{array}\right]_{l \in \bigotimes_{i=1}^k \{0,1,...,j_i\}}}{\Box(\alpha_1, ..., \alpha_j; \alpha_0) \Rightarrow \{\Box(\beta^i_1, ..., \beta^i_{j_i}; \beta^i_0)\}_{i=1}^k}$$

The hyper-sequent rule

- For a better appearance, let

$$\left(\begin{array}{lll} J = \{-j, \ldots, -1\} & K = \{1, \ldots, k\} & \\ K^{(l)} = \{y \in K \mid l(y) = 0\} & \Omega^l_K = \left\{\beta^i_{l(i)}\right\}^{l(i)\neq 0}_{i\in K} & D = \bigotimes_{i\in K}\{0, 1, \ldots, j_i\} \end{array}\right)$$

$$\frac{\left[\left|\alpha_0, \alpha_{-x} \Rightarrow \Omega^l_K\right|_{x\in J} \left|\alpha_0 \Rightarrow \beta^y_0, \Omega^l_K\right|_{y\in K^{(l)}}\right]_{l\in D}}{\Box(\alpha_1, \ldots, \alpha_j; \alpha_0) \Rightarrow \{\Box(\beta^i_1, \ldots, \beta^i_{j_i}; \beta^i_0)\}_{i\in K}}$$

- Unfold to regular-sequent rules.
  - not genuine ‘hyper’
    - only 1 seq. in the conclusion (in this only nbd rule).
  - specify for each premise (hyper-seq.) with “name” l,
    index (from $J \uplus K^{(l)}$)
    of the regular seq. that exemplifies its provability.
  - unfold to infinitely many regular seq. rules,
    with $j, k, j_1, \ldots, j_k$ and the choice of indexes as parameters.

The hyper-sequent rule

- For a better appearance, let

$$\left(\begin{array}{lll} J = \{-j, ..., -1\} & K = \{1, ..., k\} & \\ K^{(l)} = \{y \in K \mid l(y) = 0\} & \Omega_K^l = \left\{\beta^i_{l(i)}\right\}_{i \in K}^{l(i) \neq 0} & D = \bigotimes_{i \in K} \{0, 1, ..., j_i\} \end{array}\right)$$

$$\frac{\left[\left|\alpha_0, \alpha_{-x} \Rightarrow \Omega_K^l\right|_{x \in J} \left|\alpha_0 \Rightarrow \beta_0^y, \Omega_K^l\right|_{y \in K^{(l)}}\right]_{l \in D}}{\Box(\alpha_1, ..., \alpha_j; \alpha_0) \Rightarrow \{\Box(\beta_1^i, ..., \beta_{j_i}^i; \beta_0^i)\}_{i \in K}}$$

- Unfold to regular-sequent rules.
  - not genuine ‘hyper’
    - only 1 seq. in the conclusion (in this only nbd rule).
  - specify for each premise (hyper-seq.) with “name” l,
    index (from $J \uplus K^{(l)}$)
    of the regular seq. that exemplifies its provability.
  - unfold to infinitely many regular seq. rules,
    with $j, k, j_1, ..., j_k$ and the choice of indexes as parameters.

The hyper-sequent rule

- For a better appearance, let

$$\left(\begin{array}{lll} J = \{-j, ..., -1\} & K = \{1, ..., k\} & \\ K^{(I)} = \{y \in K \mid I(y) = 0\} & \Omega^I_K = \left\{\beta^i_{I(i)}\right\}^{I(i) \neq 0}_{i \in K} & D = \bigotimes_{i \in K} \{0, 1, ..., j_i\} \end{array} \right)$$

$$\frac{\left[\left|\alpha_0, \alpha_{-x} \Rightarrow \Omega^I_K\right|_{x \in J} \left|\alpha_0 \Rightarrow \beta^y_0, \Omega^I_K\right|_{y \in K^{(I)}}\right]_{I \in D}}{\Box(\alpha_1, ..., \alpha_j; \alpha_0) \Rightarrow \{\Box(\beta^i_1, ..., \beta^i_{j_i}; \beta^i_0)\}_{i \in K}}$$

- Unfold to regular-sequent rules.
  - not genuine ‘hyper’
    - only 1 seq. in the conclusion (in this only nbd rule).
  - specify for each premise (hyper-seq.) with “name” I,
    index (from $J \uplus K^{(I)}$)
    of the regular seq. that exemplifies its provability.
  - unfold to infinitely many regular seq. rules,
    with $j, k, j_1, ..., j_k$ and the choice of indexes as parameters.

The hyper-sequent rule

- For a better appearance, let

$$\left(\begin{array}{lll} J = \{-j, ..., -1\} & K = \{1, ..., k\} & \\ K^{(l)} = \{y \in K \mid l(y) = 0\} & \Omega_K^l = \left\{\beta^i_{l(i)}\right\}^{l(i)\neq 0}_{i\in K} & D = \bigotimes_{i\in K}\{0, 1, ..., j_i\} \end{array} \right)$$

$$\frac{\left[\left| \alpha_0, \alpha_{-x} \Rightarrow \Omega_K^l \right|_{x\in J} \left| \alpha_0 \Rightarrow \beta_0^y, \Omega_K^l \right|_{y\in K^{(l)}} \right]_{l\in D}}{\Box(\alpha_1, ..., \alpha_j; \alpha_0) \Rightarrow \{\Box(\beta_1^i, ..., \beta_{j_i}^i; \beta_0^i)\}_{i\in K}}$$

- Unfold to regular-sequent rules.
  - not genuine 'hyper'
    - only 1 seq. in the conclusion (in this only nbd rule).
  - specify for each premise (hyper-seq.) with "name" l,
    index (from $J \uplus K^{(l)}$)
    of the regular seq. that exemplifies its provability.
  - unfold to infinitely many regular seq. rules,
    with $j, k, j_1, ..., j_k$ and the choice of indexes as parameters.

Sequent calculus G3inl

- Start by the nbd rule of TABinl.

$$\left(\begin{array}{lll} J = \{-j, ..., -1\} & K = \{1, ..., k\} & \\ K^{(l)} = \{y \in K \mid l(y) = 0\} & \Omega^l_K = \left\{\beta^i_{l(i)}\right\}^{l(i)\neq 0}_{i \in K} & D = \bigotimes_{i \in K} \{0, 1, ..., j_i\} \end{array}\right)$$

$$\frac{\left[\left|\alpha_0, \alpha_{-x} \Rightarrow \Omega^l_K\right|_{x \in J} \left|\alpha_0 \Rightarrow \beta^y_0, \Omega^l_K\right|_{y \in K^{(l)}}\right]_{l \in D}}{\Box(\alpha_1, ..., \alpha_j; \alpha_0) \Rightarrow \{\Box(\beta^i_1, ..., \beta^i_{j_i}; \beta^i_0)\}_{i \in K}}$$

- Choice of indexes presented by a function f on names:

$$\forall l \in D \begin{cases} f(l) = x \in J & \text{if premise l is } \vdash \alpha_0, \alpha_{-x} \Rightarrow \Omega^l_K, \\ f(l) = y \in K^{(l)} & \text{if premise l is } \vdash \alpha_0 \Rightarrow \beta^y_0, \Omega^l_K. \end{cases}$$

We write f_l instead of $f(l)$ for a better appearance...

- $f : D \mapsto J \cup K$ s.t.

(adequacy) $(\forall l \in D)(f_l \in K$ implies $f_l \in K^{(l)}$, e.g., $l(f_l) = 0)$.

- Start by the nbd rule of TABinl.

$$\left(\begin{array}{lll} J = \{-j, ..., -1\} & K = \{1, ..., k\} & \\ K^{(l)} = \{y \in K \mid l(y) = 0\} & \Omega^l_K = \left\{\beta^i_{l(i)}\right\}^{l(i) \neq 0}_{i \in K} & D = \bigotimes_{i \in K} \{0, 1, ..., j_i\} \end{array} \right)$$

$$\frac{\left[\left| \alpha_0, \alpha_{-x} \Rightarrow \Omega^l_K \right|_{x \in J} \left| \alpha_0 \Rightarrow \beta^y_0, \Omega^l_K \right|_{y \in K^{(l)}} \right]_{l \in D}}{\Box(\alpha_1, ..., \alpha_j; \alpha_0) \Rightarrow \{\Box(\beta^i_1, ..., \beta^i_{j_i}; \beta^i_0)\}_{i \in K}}$$

- Choice of indexes presented by a function f on names:

$$\forall l \in D \begin{cases} f(l) = x \in J & \text{if premise l is } \vdash \alpha_0, \alpha_{-x} \Rightarrow \Omega^l_K, \\ f(l) = y \in K^{(l)} & \text{if premise l is } \vdash \alpha_0 \Rightarrow \beta^y_0, \Omega^l_K. \end{cases}$$

We write f_l instead of $f(l)$ for a better appearance...

- $f : D \mapsto J \cup K$ s.t.

(adequacy) $(\forall l \in D)(f_l \in K$ implies $f_l \in K^{(l)}$, e.g., $l(f_l) = 0)$.

Sequent calculus G3inl

- Start by the nbd rule of TABinl.

$$\left(\begin{array}{lll} J = \{-j, ..., -1\} & K = \{1, ..., k\} & \\ K^{(l)} = \{y \in K \mid l(y) = 0\} & \Omega^l_K = \left\{\beta^i_{l(i)}\right\}^{l(i)\neq 0}_{i\in K} & D = \bigotimes_{i\in K}\{0, 1, ..., j_i\} \end{array} \right)$$

$$\frac{\left[\left|\alpha_0, \alpha_{-x} \Rightarrow \Omega^l_K\right|_{x\in J} \left|\alpha_0 \Rightarrow \beta^y_0, \Omega^l_K\right|_{y\in K^{(l)}}\right]_{l\in D}}{\Box(\alpha_1, ..., \alpha_j; \alpha_0) \Rightarrow \{\Box(\beta^i_1, ..., \beta^i_{j_i}; \beta^i_0)\}_{i\in K}}$$

- Choice of indexes presented by a function f on names:

$$\forall l \in D \begin{cases} f(l) = x \in J & \text{if premise l is } \vdash \alpha_0, \alpha_{-x} \Rightarrow \Omega^l_K, \\ f(l) = y \in K^{(l)} & \text{if premise l is } \vdash \alpha_0 \Rightarrow \beta^y_0, \Omega^l_K. \end{cases}$$

  We write f_l instead of $f(l)$ for a better appearance...

- $f : D \mapsto J \cup K$ s.t.

  (adequacy) $(\forall l \in D)(f_l \in K$ implies $f_l \in K^{(l)}$, e.g., $l(f_l) = 0)$.

Sequent calculus G3inl

- Group premises w.r.t. f-images of their names.

$$\left(\begin{array}{lll} J = \{-j, ..., -1\} & K = \{1, ..., k\} & \\ K^{(l)} = \{y \in K \mid l(y) = 0\} & \Omega_K^l = \left\{\beta^i_{l(i)}\right\}^{l(i) \neq 0}_{i \in K} & D = \bigotimes_{i \in K} \{0, 1, ..., j_i\} \end{array} \right)$$

$$\frac{\left[\alpha_0, \alpha_{-f_l} \Rightarrow \Omega_K^l\right]^{f_l \in J}_{l \in D} \left[\alpha_0 \Rightarrow \beta_0^{f_l}, \Omega_K^l\right]^{f_l \in K^{(l)}}_{l \in D}}{\Box(\alpha_1, ..., \alpha_j; \alpha_0) \Rightarrow \{\Box(\beta_1^i, ..., \beta_{j_i}^i; \beta_0^i)\}_{i \in K}}$$

- Choice of indexes presented by a function f on names:

$$\forall l \in D \begin{cases} f(l) = x \in J & \text{if premise l is } \vdash \alpha_0, \alpha_{-x} \Rightarrow \Omega_K^l, \\ f(l) = y \in K^{(l)} & \text{if premise l is } \vdash \alpha_0 \Rightarrow \beta_0^y, \Omega_K^l. \end{cases}$$

  We write f_l instead of $f(l)$ for a better appearance...

- $f : D \mapsto J \cup K$ s.t.

  (adequacy) $(\forall l \in D)(f_l \in K$ implies $f_l \in K^{(l)}$, e.g., $l(f_l) = 0)$.

Sequent calculus G3inl

- Group premises w.r.t. f-images of their names.

$$\left(\begin{array}{lll} J = \{-j, \ldots, -1\} & K = \{1, \ldots, k\} & \\ K^{(l)} = \{y \in K \mid l(y) = 0\} & \Omega^l_K = \left\{\beta^i_{l(i)}\right\}^{l(i)\neq 0}_{i\in K} & D = \bigotimes_{i\in K}\{0, 1, \ldots, j_i\} \end{array} \right)$$

$$\frac{\left[\alpha_0, \alpha_{-f_l} \Rightarrow \Omega^l_K\right]^{f_l\in J}_{l\in D} \left[\alpha_0 \Rightarrow \beta^{f_l}_0, \Omega^l_K\right]^{f_l\in K^{(l)}}_{l\in D}}{\Pi, \Box(\alpha_1, \ldots, \alpha_j; \alpha_0) \Rightarrow \{\Box(\beta^i_1, \ldots, \beta^i_{j_i}; \beta^i_0)\}_{i\in K}, \Sigma}$$

- In multi-set based G3, built-in weakening is necessary.
- This is $\left(\Box^{j,k,f}_{\langle j_1,\ldots,j_k\rangle}\right)$, nbd rule with parameters $j, k, j_1, \ldots, j_k, f$.
- It respects the proper sub-formula property (no built-in contraction).
- G3inl is G3cp (in the language of INL) extended by all $\left(\Box^{j,k,f}_{\langle j_1,\ldots,j_k\rangle}\right)$ where f is adequate w.r.t. its other parameters.

Sequent calculus G3inl

- Group premises w.r.t. f-images of their names.

$$\left(\begin{array}{lll} J = \{-j, ..., -1\} & K = \{1, ..., k\} & \\ K^{(l)} = \{y \in K \mid l(y) = 0\} & \Omega^l_K = \left\{\beta^i_{l(i)}\right\}^{l(i) \neq 0}_{i \in K} & D = \bigotimes_{i \in K} \{0, 1, ..., j_i\} \end{array}\right)$$

$$\frac{\left[\alpha_0, \alpha_{-f_l} \Rightarrow \Omega^l_K\right]^{f_l \in J}_{l \in D} \left[\alpha_0 \Rightarrow \beta^{f_l}_0, \Omega^l_K\right]^{f_l \in K^{(l)}}_{l \in D}}{\Pi, \Box(\alpha_1, ..., \alpha_j; \alpha_0) \Rightarrow \{\Box(\beta^i_1, ..., \beta^i_{j_i}; \beta^i_0)\}_{i \in K}, \Sigma}$$

- In multi-set based G3, built-in weakening is necessary.
- This is $\left(\Box^{j,k,f}_{\langle j_1, ..., j_k \rangle}\right)$, nbd rule with parameters $j, k, j_1, ..., j_k, f$.
- It respects the proper sub-formula property (no built-in contraction).
- G3inl is G3cp (in the language of INL) extended by all $\left(\Box^{j,k,f}_{\langle j_1, ..., j_k \rangle}\right)$ where f is adequate w.r.t. its other parameters.

Sequent calculus G3inl

- Group premises w.r.t. f-images of their names.

$$\left(\begin{array}{lll} J = \{-j, ..., -1\} & K = \{1, ..., k\} & \\ K^{(l)} = \{y \in K \mid l(y) = 0\} & \Omega^l_K = \left\{\beta^i_{l(i)}\right\}^{l(i)\neq 0}_{i\in K} & D = \bigotimes_{i\in K}\{0, 1, ..., j_i\} \end{array}\right)$$

$$\frac{\left[\alpha_0, \alpha_{-f_l} \Rightarrow \Omega^l_K\right]^{f_l\in J}_{l\in D} \left[\alpha_0 \Rightarrow \beta^{f_l}_0, \Omega^l_K\right]^{f_l\in K^{(l)}}_{l\in D}}{\Pi, \Box(\alpha_1, ..., \alpha_j; \alpha_0) \Rightarrow \{\Box(\beta^i_1, ..., \beta^i_{j_i}; \beta^i_0)\}_{i\in K}, \Sigma}$$

- In multi-set based G3, built-in weakening is necessary.
- This is $\left(\Box^{j,k,f}_{\langle j_1,...,j_k\rangle}\right)$, nbd rule with parameters $j, k, j_1, ..., j_k, f$.
- It respects the proper sub-formula property (no built-in contraction).
- G3inl is G3cp (in the language of INL) extended by all $\left(\Box^{j,k,f}_{\langle j_1,...,j_k\rangle}\right)$ where f is adequate w.r.t. its other parameters.

Sequent calculus G3inl

- G3inl
  - admits Weakening, Contraction, and Cut;
  - is sound and complete (via equivalent with Hinl).
  - enjoys the sub-formula property,
    supports mechanical proof-search.
- A splitting version of G3inl leads to a constructive proof of INL's Lyndon interpolation theorem.

Sequent calculus G3inl

- G3inl
  - admits Weakening, Contraction, and Cut;
  - is sound and complete (via equivalent with Hinl).
  - enjoys the sub-formula property,
    supports mechanical proof-search.
- A splitting version of G3inl leads to a constructive proof of INL's Lyndon interpolation theorem.

Sequent calculus G3inl

- G3inl
  - admits Weakening, Contraction, and Cut;
  - is sound and complete (via equivalent with Hinl).
  - enjoys the sub-formula property,
    supports mechanical proof-search.
- A splitting version of G3inl leads to a constructive proof of INL's Lyndon interpolation theorem.

Lyndon interpolation theorem

Lyndon interpolation of INL

- Lydon interpolation theorem:

  (Let $\mathcal{V}^+(\alpha)$/$\mathcal{V}^-(\alpha)$ denotes positive/negative atoms in α)

  If INL $\vdash \phi \to \psi$, then there is a formula θ s.t.:
  - $\mathcal{V}^{\pm}(\theta) \subseteq \mathcal{V}^{\pm}(\phi) \cap \mathcal{V}^{\pm}(\psi)$
  - INL $\vdash \phi \to \theta$ and INL $\vdash \theta \to \psi$.

    (a 'polarized' Craig interpolation)

- A general form on sequents:

  If G3inl $\vdash \Pi_L, \Pi_R \Rightarrow \Sigma_L, \Sigma_R$, then there is a formula θ s.t.:
  - $\mathcal{V}^{\pm}(\theta) \subseteq (\mathcal{V}^{\mp}(\Pi_R, \Sigma_L)) \cap (\mathcal{V}^{\pm}(\Pi_L, \Sigma_R))$
  - G3inl $\vdash \Pi_L \Rightarrow \Sigma_L, \theta$ and G3inl $\vdash \theta, \Pi_R \Rightarrow \Sigma_R$.

- The method of splitting sequent calculus (K. Schütte) works.

Lyndon interpolation of INL

- Lydon interpolation theorem:

  (Let $\mathcal{V}^{+}(\alpha)/\mathcal{V}^{-}(\alpha)$ denotes positive/negative atoms in α)

  If INL $\vdash \phi \to \psi$, then there is a formula θ s.t.:
  - $\mathcal{V}^{\pm}(\theta) \subseteq \mathcal{V}^{\pm}(\phi) \cap \mathcal{V}^{\pm}(\psi)$
  - INL $\vdash \phi \to \theta$ and INL $\vdash \theta \to \psi$.

  (a 'polarized' Craig interpolation)
- A general form on sequents:

  If G3inl $\vdash \Pi_L, \Pi_R \Rightarrow \Sigma_L, \Sigma_R$, then there is a formula θ s.t.:
  - $\mathcal{V}^{\pm}(\theta) \subseteq (\mathcal{V}^{\mp}(\Pi_R, \Sigma_L)) \cap (\mathcal{V}^{\pm}(\Pi_L, \Sigma_R))$
  - G3inl $\vdash \Pi_L \Rightarrow \Sigma_L, \theta$ and G3inl $\vdash \theta, \Pi_R \Rightarrow \Sigma_R$.
- The method of splitting sequent calculus (K. Schütte) works.

Lyndon interpolation of INL

- Lydon interpolation theorem:

  (Let $\mathcal{V}^+(\alpha)/\mathcal{V}^-(\alpha)$ denotes positive/negative atoms in α)

  If INL $\vdash \phi \rightarrow \psi$, then there is a formula θ s.t.:

  - $\mathcal{V}^{\pm}(\theta) \subseteq \mathcal{V}^{\pm}(\phi) \cap \mathcal{V}^{\pm}(\psi)$
  - INL $\vdash \phi \rightarrow \theta$ and INL $\vdash \theta \rightarrow \psi$.

  (a 'polarized' Craig interpolation)

- A general form on sequents:

  If G3inl $\vdash \Pi_L, \Pi_R \Rightarrow \Sigma_L, \Sigma_R$, then there is a formula θ s.t.:

  - $\mathcal{V}^{\pm}(\theta) \subseteq (\mathcal{V}^{\mp}(\Pi_R, \Sigma_L)) \cap (\mathcal{V}^{\pm}(\Pi_L, \Sigma_R))$
  - G3inl $\vdash \Pi_L \Rightarrow \Sigma_L, \theta$ and G3inl $\vdash \theta, \Pi_R \Rightarrow \Sigma_R$.

- The method of splitting sequent calculus (K. Schütte) works.

- A splitting version of G3inl
  - Antecedent and succedent of each sequent split into two parts (left | right), like $\Gamma_L|\Gamma_R \Rightarrow \Delta_L|\Delta_R$.
  - For each rule, given an arbitrary splitting conclusion,
    depending on locations of the principals,
    the splitting calculus should offer a rule
    with splitting premises
    where actives respect polarities,
    and also specify a construction
    of the conclusion's interpolant
    based on assumed interpolants of premises.

Lyndon interpolation of INL

- A splitting version of G3inl
  - Antecedent and succedent of each sequent split into two parts (left | right), like $\Gamma_L|\Gamma_R \Rightarrow \Delta_L|\Delta_R$.
  - For each rule, given an arbitrary splitting conclusion,
    depending on locations of the principals,
    the splitting calculus should offer a rule
    with splitting premises
    where actives respect polarities,
    and also specify a construction
    of the conclusion's interpolant
    based on assumed interpolants of premises.

Lyndon interpolation of INL

- For the nbe rule $(\Box^{j,k,f}_{\langle j_1,\ldots,j_k\rangle})$:

$$\frac{\left[\alpha_0, \alpha_{-f_l} \Rightarrow \Omega^l_K\right]^{f_l\in J}_{l\in D} \left[\alpha_0 \Rightarrow \beta^{f_l}_0, \Omega^l_K\right]^{f_l\in K}_{l\in D}}{\Pi, \Box(\alpha_1,\ldots,\alpha_j;\alpha_0) \Rightarrow \{\Box(\beta^i_1,\ldots,\beta^i_{j_i};\beta^i_0)\}_{i\in K}, \Sigma}$$

  - Weakening Π, Σ never matter.
  - Assume the negative principal $\Box(\alpha_1,\ldots,\alpha_j;\alpha_0)$ goes left,

    the case that it goes right is quite similar.
  - Recall: $J = \{-j,\ldots,-1\}$, $K = \{1,\ldots,k\}$, $D = \bigotimes_{i\in K}\{0,1,\ldots,j_i\}$.
  - W.l.o.g., assume that positive principals split to $\{\Box(\beta^i_1,\ldots,\beta^i_{j_i})\}_{i\in L} | \{\Box(\beta^i_1,\ldots,\beta^i_{j_i})\}_{i\in R}$ (where $L \uplus R = K$).
  - Also let $D_L = \bigotimes_{i\in L}\{0,1,\ldots,j_i\}$, $D_R = \bigotimes_{i\in R}\{0,1,\ldots,j_i\}$.
    For $l = \langle l(1),\ldots,l(k)\rangle \in D$, let $l_L := l\restriction L \in D_L$, and $l_R := l\restriction R \in D_R$.
    Let $\mathbf{0}^R$ be a list of 0's with index-set R, similar for $\mathbf{0}^L$.

    Denote the index-disjoint merge of lists M and N by $M \circ N$.

Lyndon interpolation of INL

- For the nbe rule $(\Box^{j,k,f}_{\langle j_1,\dots,j_k\rangle})$:

$$\frac{\left[\alpha_0, \alpha_{-f_l} \Rightarrow \Omega^l_K\right]^{f_l \in J}_{l \in D} \left[\alpha_0 \Rightarrow \beta^{f_l}_0, \Omega^l_K\right]^{f_l \in K}_{l \in D}}{\Pi, \Box(\alpha_1, \dots, \alpha_j; \alpha_0) \Rightarrow \{\Box(\beta^i_1, \dots, \beta^i_{j_i}; \beta^i_0)\}_{i \in K}, \Sigma}$$

  - Weakening Π, Σ never matter.
  - Assume the negative principal $\Box(\alpha_1, \dots, \alpha_j; \alpha_0)$ goes left,

    the case that it goes right is quite similar.
  - Recall: $J = \{-j, \dots, -1\}$, $K = \{1, \dots, k\}$, $D = \bigotimes_{i \in K}\{0, 1, \dots, j_i\}$.
  - W.l.o.g., assume that positive principals split to $\{\Box(\beta^i_1, \dots, \beta^i_{j_i})\}_{i \in L} | \{\Box(\beta^i_1, \dots, \beta^i_{j_i})\}_{i \in R}$ (where $L \uplus R = K$).
  - Also let $D_L = \bigotimes_{i \in L}\{0, 1, \dots, j_i\}$, $D_R = \bigotimes_{i \in R}\{0, 1, \dots, j_i\}$.

    For $l = \langle l(1), \dots, l(k)\rangle \in D$, let $l_L := l \upharpoonright L \in D_L$, and $l_R := l \upharpoonright R \in D_R$.

    Let $\mathbf{0}^R$ be a list of 0's with index-set R, similar for $\mathbf{0}^L$.

    Denote the index-disjoint merge of lists M and N by $M \circ N$.

Lyndon interpolation of INL

- For the nbe rule $(\Box^{j,k,f}_{\langle j_1,\dots,j_k\rangle})$:

$$\frac{\left[\alpha_0, \alpha_{-f_l} \Rightarrow \Omega^l_K\right]^{f_l\in J}_{l\in D} \quad \left[\alpha_0 \Rightarrow \beta^{f_l}_0, \Omega^l_K\right]^{f_l\in K}_{l\in D}}{\Pi, \Box(\alpha_1, \dots, \alpha_j; \alpha_0) \Rightarrow \{\Box(\beta^i_1, \dots, \beta^i_{j_i}; \beta^i_0)\}_{i\in K}, \Sigma}$$

  - Weakening Π, Σ never matter.
  - Assume the negative principal $\Box(\alpha_1, \dots, \alpha_j; \alpha_0)$ goes left, the case that it goes right is quite similar.
  - Recall: $J = \{-j, \dots, -1\}$, $K = \{1, \dots, k\}$, $D = \bigotimes_{i\in K}\{0, 1, \dots, j_i\}$.
  - W.l.o.g., assume that positive principals split to $\{\Box(\beta^i_1, \dots, \beta^i_{j_i})\}_{i\in L} | \{\Box(\beta^i_1, \dots, \beta^i_{j_i})\}_{i\in R}$ (where $L \uplus R = K$).
  - Also let $D_L = \bigotimes_{i\in L}\{0, 1, \dots, j_i\}$, $D_R = \bigotimes_{i\in R}\{0, 1, \dots, j_i\}$. For $l = \langle l(1), \dots, l(k)\rangle \in D$, let $l_L := l \restriction L \in D_L$, and $l_R := l \restriction R \in D_R$. Let $\mathbf{0}^R$ be a list of 0's with index-set R, similar for $\mathbf{0}^L$.

    Denote the index-disjoint merge of lists M and N by $M \circ N$.

Lyndon interpolation of INL

- For the nbe rule ($\Box^{j,k,f}_{\langle j_1,\dots,j_k\rangle}$):

$$\frac{\left[\alpha_0, \alpha_{-f_l} \Rightarrow \Omega^l_K\right]^{f_l \in J}_{l \in D} \left[\alpha_0 \Rightarrow \beta^{f_l}_0, \Omega^l_K\right]^{f_l \in K}_{l \in D}}{\Pi, \Box(\alpha_1, \dots, \alpha_j; \alpha_0) \Rightarrow \{\Box(\beta^i_1, \dots, \beta^i_{j_i}; \beta^i_0)\}_{i \in K}, \Sigma}$$

  - Weakening Π, Σ never matter.
  - Assume the negative principal $\Box(\alpha_1, \dots, \alpha_j; \alpha_0)$ goes left,

    the case that it goes right is quite similar.
  - Recall: $J = \{-j, \dots, -1\}$, $K = \{1, \dots, k\}$, $D = \bigotimes_{i \in K}\{0, 1, \dots, j_i\}$.
  - W.l.o.g., assume that positive principals split to $\{\Box(\beta^i_1, \dots, \beta^i_{j_i})\}_{i \in L} | \{\Box(\beta^i_1, \dots, \beta^i_{j_i})\}_{i \in R}$ (where $L \uplus R = K$).
  - Also let $D_L = \bigotimes_{i \in L}\{0, 1, \dots, j_i\}$, $D_R = \bigotimes_{i \in R}\{0, 1, \dots, j_i\}$.
    For $I = \langle I(1), \dots, I(k)\rangle \in D$, let $I_L := I \restriction L \in D_L$, and $I_R := I \restriction R \in D_R$.
    Let $\mathbf{0}^R$ be a list of 0's with index-set R, similar for $\mathbf{0}^L$.

    Denote the index-disjoint merge of lists M and N by $M \circ N$.

Lyndon interpolation of INL

- When the only negative principal goes left:

$$\frac{\left[\alpha_0, \alpha_{-f_l} \Rightarrow \Omega^l_K\right]^{f_l \in J}_{l \in D} \quad \left[\alpha_0 \Rightarrow \beta^{f_l}_0, \Omega^l_K\right]^{f_l \in K}_{l \in D}}{\Box(\alpha_1, \ldots, \alpha_j; \alpha_0)| \Rightarrow \{\Box(\beta^i_1, \ldots, \beta^i_{j_i}; \beta^i_0)\}_{i \in L} | \{\Box(\beta^i_1, \ldots, \beta^i_{j_i}; \beta^i_0)\}_{i \in R}}$$

  - For premise with name $l \in D$, choose split premise:
    - $\alpha_0, \alpha_{-f_l}| \Rightarrow \Omega^{l_L}_L | \Omega^{l_R}_R$, when $f_l \in J$;
    - $\alpha_0| \Rightarrow \Omega^{l_L}_L, \beta^{f_l}_0 | \Omega^{l_R}_R$, when $f_l \in L \subseteq K$;
    - $\alpha_0| \Rightarrow \Omega^{l_L}_L | \beta^{f_l}_0, \Omega^{l_R}_R$, when $f_l \in R \subseteq K$.

    and denote its assumed interpolant by θ_l.
  - The desired interpolant for the conclusion is

$$\bigvee_{M \in D_L}^{f_{M \circ \mathbf{0}^R} \in R} \Box \left(\left\{\theta_{M \circ N}\right\}^{f_{M \circ N} \in J \cup L}_{N \in D_R} ; \bigwedge_{N \in D_R}^{f_{M \circ N} \in R} \theta_{M \circ N} \right)$$

Lyndon interpolation of INL

- When the only negative principal goes left:

$$\frac{\left[\alpha_0, \alpha_{-f_l} \Rightarrow \Omega_K^l\right]_{l\in D}^{f_l\in J} \left[\alpha_0 \Rightarrow \beta_0^{f_l}, \Omega_K^l\right]_{l\in D}^{f_l\in K}}{\Box(\alpha_1, ..., \alpha_j; \alpha_0)| \Rightarrow \{\Box(\beta_1^i, ..., \beta_{j_i}^i; \beta_0^i)\}_{i\in L}|\{\Box(\beta_1^i, ..., \beta_{j_i}^i; \beta_0^i)\}_{i\in R}}$$

  - For premise with name $l \in D$, choose split premise:
    - $\alpha_0, \alpha_{-f_l}| \Rightarrow \Omega_L^{l_L}|\Omega_R^{l_R}$, when $f_l \in J$;
    - $\alpha_0| \Rightarrow \Omega_L^{l_L}, \beta_0^{f_l}|\Omega_R^{l_R}$, when $f_l \in L \subseteq K$;
    - $\alpha_0| \Rightarrow \Omega_L^{l_L}|\beta_0^{f_l}, \Omega_R^{l_R}$, when $f_l \in R \subseteq K$.

    and denote its assumed interpolant by θ_l.
  - The desired interpolant for the conclusion is

$$\bigvee_{M\in D_L}^{f_{M\circ 0^R}\in R} \Box\left(\left\{\theta_{M\circ N}\right\}_{N\in D_R}^{f_{M\circ N}\in J\cup L}; \bigwedge_{N\in D_R}^{f_{M\circ N}\in R} \theta_{M\circ N}\right)$$

Lyndon interpolation of INL

- When the only negative principal goes left:

$$\frac{\left[\alpha_0, \alpha_{-f_l} \Rightarrow \Omega^l_K\right]^{f_l \in J}_{l \in D} \quad \left[\alpha_0 \Rightarrow \beta_0^{f_l}, \Omega^l_K\right]^{f_l \in K}_{l \in D}}{\Box(\alpha_1, \ldots, \alpha_j; \alpha_0)| \Rightarrow \{\Box(\beta^i_1, \ldots, \beta^i_{j_i}; \beta^i_0)\}_{i \in L} | \{\Box(\beta^i_1, \ldots, \beta^i_{j_i}; \beta^i_0)\}_{i \in R}}$$

  - For premise with name $l \in D$, choose split premise:
    - $\alpha_0, \alpha_{-f_l}| \Rightarrow \Omega^{l_L}_L | \Omega^{l_R}_R$, when $f_l \in J$;
    - $\alpha_0| \Rightarrow \Omega^{l_L}_L, \beta_0^{f_l} | \Omega^{l_R}_R$, when $f_l \in L \subseteq K$;
    - $\alpha_0| \Rightarrow \Omega^{l_L}_L | \beta_0^{f_l}, \Omega^{l_R}_R$, when $f_l \in R \subseteq K$.

    and denote its assumed interpolant by θ_l.
  - The desired interpolant for the conclusion is

$$\bigvee_{M \in D_L}^{f_{M \circ \mathbf{0}^R} \in R} \Box \left(\left\{\theta_{M \circ N}\right\}^{f_{M \circ N} \in J \cup L}_{N \in D_R} ; \bigwedge_{N \in D_R}^{f_{M \circ N} \in R} \theta_{M \circ N} \right)$$

Lyndon interpolation of INL

- When the only negative principal goes left:

$$\frac{\left[\alpha_0, \alpha_{-f_l} \Rightarrow \Omega^l_K\right]^{f_l \in J}_{l \in D} \left[\alpha_0 \Rightarrow \beta^{f_l}_0, \Omega^l_K\right]^{f_l \in K}_{l \in D}}{\Box(\alpha_1, \ldots, \alpha_j; \alpha_0)| \Rightarrow \{\Box(\beta^i_1, \ldots, \beta^i_{j_i}; \beta^i_0)\}_{i \in L} | \{\Box(\beta^i_1, \ldots, \beta^i_{j_i}; \beta^i_0)\}_{i \in R}}$$

  - For premise with name $l \in D$, choose split premise:
    - $\alpha_0, \alpha_{-f_l}| \Rightarrow \Omega^{l_L}_L | \Omega^{l_R}_R$, when $f_l \in J$;
    - $\alpha_0| \Rightarrow \Omega^{l_L}_L, \beta^{f_l}_0 | \Omega^{l_R}_R$, when $f_l \in L \subseteq K$;
    - $\alpha_0| \Rightarrow \Omega^{l_L}_L | \beta^{f_l}_0, \Omega^{l_R}_R$, when $f_l \in R \subseteq K$.

    and denote its assumed interpolant by θ_l.
  - The desired interpolant for the conclusion is

$$\bigvee_{M \in D_L}^{f_{M \circ 0^R} \in R} \Box\left(\left\{\theta_{M \circ N}\right\}^{f_{M \circ N} \in J \cup L}_{N \in D_R}; \bigwedge_{N \in D_R}^{f_{M \circ N} \in R} \theta_{M \circ N}\right)$$

Lyndon interpolation of INL

- When the only negative principal goes left:

$$\frac{\left[\alpha_0, \alpha_{-f_l} \Rightarrow \Omega^l_K\right]^{f_l \in J}_{l \in D} \left[\alpha_0 \Rightarrow \beta^{f_l}_0, \Omega^l_K\right]^{f_l \in K}_{l \in D}}{\Box(\alpha_1, \ldots, \alpha_j; \alpha_0)| \Rightarrow \{\Box(\beta^i_1, \ldots, \beta^i_{j_i}; \beta^i_0)\}_{i \in L} | \{\Box(\beta^i_1, \ldots, \beta^i_{j_i}; \beta^i_0)\}_{i \in R}}$$

  - For premise with name $l \in D$, choose split premise:
    - $\alpha_0, \alpha_{-f_l}| \Rightarrow \Omega^{l_L}_L | \Omega^{l_R}_R$, when $f_l \in J$;
    - $\alpha_0| \Rightarrow \Omega^{l_L}_L, \beta^{f_l}_0 | \Omega^{l_R}_R$, when $f_l \in L \subseteq K$;
    - $\alpha_0| \Rightarrow \Omega^{l_L}_L | \beta^{f_l}_0, \Omega^{l_R}_R$, when $f_l \in R \subseteq K$.

    and denote its assumed interpolant by θ_l.
  - The desired interpolant for the conclusion is

$$\bigvee_{M \in D_L}^{f_{M \circ \mathbf{0}^R} \in R} \Box \left(\left\{ \theta_{M \circ N} \right\}^{f_{M \circ N} \in J \cup L}_{N \in D_R} ; \bigwedge_{N \in D_R}^{f_{M \circ N} \in R} \theta_{M \circ N} \right)$$

- When the only negative principal goes right:

$$\frac{\left[\alpha_0, \alpha_{-f_l} \Rightarrow \Omega^l_K\right]^{f_l \in J}_{l \in D} \left[\alpha_0 \Rightarrow \beta^{f_l}_0, \Omega^l_K\right]^{f_l \in K}_{l \in D}}{|\Box(\alpha_1, \ldots, \alpha_j; \alpha_0) \Rightarrow \{\Box(\beta^i_1, \ldots, \beta^i_{j_i}; \beta^i_0)\}_{i \in L} | \{\Box(\beta^i_1, \ldots, \beta^i_{j_i}; \beta^i_0)\}_{i \in R}}$$

  - For premise with name $l \in D$, choose split premise:
    - $|\alpha_0, \alpha_{-f_l} \Rightarrow \Omega^{l_L}_L | \Omega^{l_R}_R$, when $f_l \in J$;
    - $|\alpha_0 \Rightarrow \Omega^{l_L}_L, \beta^{f_l}_0 | \Omega^{l_R}_R$, when $f_l \in L \subseteq K$;
    - $|\alpha_0 \Rightarrow \Omega^{l_L}_L | \beta^{f_l}_0, \Omega^{l_R}_R$, when $f_l \in R \subseteq K$.

    and denote its assumed interpolant by θ_l.
  - The desired interpolant for the conclusion is

$$\neg \bigvee^{f_{0L \circ N} \in L}_{N \in D_R} \Box \left(\left\{ \neg \theta_{M \circ N} \right\}^{f_{M \circ N} \in J \cup R}_{M \in D_L} ; \bigwedge^{f_{M \circ N} \in L}_{M \in D_L} \neg \theta_{M \circ N} \right)$$

- Future directions

Future Directions

- More properties of the nbd function
  - leads to genuine hyper-sequent ?
- Modification on language
  - bound on number of instances
  - infinite language (conjunctions, disjunctions, instances).
- Nabla operator v.s. $\Box(\alpha_1, ..., \alpha_j; \alpha_1 \vee ... \vee \alpha_j)$.
- Uniform interpolation
  - the method of R. Iemhoff seems to work.

Future Directions

- More properties of the nbd function
  - leads to genuine hyper-sequent ?
- Modification on language
  - bound on number of instances
  - infinite language (conjunctions, disjunctions, instances).
- Nabla operator v.s. $\Box(\alpha_1, ..., \alpha_j; \alpha_1 \vee ... \vee \alpha_j)$.
- Uniform interpolation
  - the method of R. Iemhoff seems to work.

Future Directions

- More properties of the nbd function
  - leads to genuine hyper-sequent ?
- Modification on language
  - bound on number of instances
  - infinite language (conjunctions, disjunctions, instances).
- Nabla operator v.s. $\Box(\alpha_1, ..., \alpha_j; \alpha_1 \vee ... \vee \alpha_j)$.
- Uniform interpolation
  - the method of R. Iemhoff seems to work.

Future Directions

- More properties of the nbd function
  - leads to genuine hyper-sequent ?
- Modification on language
  - bound on number of instances
  - infinite language (conjunctions, disjunctions, instances).
- Nabla operator v.s. $\Box(\alpha_1, ..., \alpha_j; \alpha_1 \vee ... \vee \alpha_j)$.
- Uniform interpolation
  - the method of R. Iemhoff seems to work.

Future Directions

- More properties of the nbd function
  - leads to genuine hyper-sequent ?
- Modification on language
  - bound on number of instances
  - infinite language (conjunctions, disjunctions, instances).
- Nabla operator v.s. $\Box(\alpha_1, ..., \alpha_j; \alpha_1 \vee ... \vee \alpha_j)$.
- Uniform interpolation
  - the method of R. Iemhoff seems to work.

- Thanks !